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# Quantum Theory of Elementary Processes

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In modern physics, one of the greatest divides is that between space–time and quantum fields, as the fiber bundle of the Standard Model indicates. However, on operational grounds the fields and space–time are not very different. To describe a field in an experimental region we have to assign coordinates to the points of that region in order to speak of "when" and "where" of the field itself. But to operationally study the topology and to coordinatize the region of space–time, the use of radars (to send and receive electromagnetic signals) is required. Thus the description of fields (or, rather, processes) and the description of space–time are indistinguishable at the fundamental level. Moreover, classical general relativity already says—albeit preserving the fiber bundle structure—that space–time and matter are intimately related. All this indicates that a new theory of elementary processes (out of which all the usual processes of creation, annihilation, and propagation, and consequently the topology of space–time itself would be constructed) has to be devised. In this review the foundations of such a finite, discrete, algebraic, quantum theory are summarized. The theory is then applied to the description of spin-1/2 quanta of the Standard Model.

**KEY WORDS:** quantum theory; elementary processes.

## 1. INTRODUCTION

> *. . . One can give good reasons why reality cannot at all be represented by a continuous field. From the quantum phenomena it appears to follow with certainty that a finite system of finite energy can be completely described by a finite set of numbers (quantum numbers). This does not seem to be in accordance with a continuum theory, and must lead to an attempt to find a purely algebraic theory for the description of reality. But nobody knows how to obtain the basis of such a theory.*
>
> Albert Einstein, 1955

There are essentially three major areas of physics that have been puzzling me for the last several years: possibility of quantum-field – space–time unification beyond the Standard Model, its experimental tests, and the cosmological constant problem. All of them are very closely related, and reflect our, physicists', belief that Nature at its deepest and most fundamental level is simple, is governed by

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simple universal laws, and that there must be a theory that could describe, explain, and unify all the intricacies of the world around us in one beautiful and elegant scheme. I do, however, believe that no such “final” theory of Nature can ever be found, at least not the one based on the mathematical axiomatic method, as Goedel’s theorems indicate (Appendix A).

The quest for unification originated in my numerous conversations with David Finkelstein, who had always emphasized the importance of algebraic simplicity in physics. This simplicity is exactly what mathematicians mean when they speak of simple groups and algebras. In physics, algebraic simplicity is the symbol of unity and beauty. For example, in classical mechanics, if time couples into space (as is expressed by Galilean transformations), then there should be—on the grounds of simplicity—a coupling of space into time, and that’s exactly what Lorentz transformations establish.

In modern physics, one of the greatest divides is that between space–time and quantum fields, as the fiber bundle of the Standard Model indicates. However, on operational grounds the fields and space–time are not very different. To describe a field in an experimental region we have to assign coordinates to the points of that region in order to speak of “when” and “where” of the field itself. But to operationally study the topology and to coordinatize the region of space–time, the use of radars (to send and receive electromagnetic singals) is required. Thus the description of fields (or, rather, processes) and the description of space–time are indistinguishable at the fundamental level. Moreover, classical general relativity already says—albeit preserving the fiber bundle structure—that space–time and matter are intimately related. All this indicates that a new theory of elementary processes (out of which all the usual processes of creation, annihilation, and propagation, and consequently the topology of space–time itself would be constructed) has to be devised.

In this paper I present the foundations of such a finite, discrete, algebraic, quantum theory, and apply it to the description of spin-1/2 quanta of the Standard Model. The basic principle of theory can be summarized as follows:

1) *Operationality*. The statements of physics have the form: *If we do so-and-so, we will find such-and-such*. The primary element of the theory is thus a *process* (called *operation* when driven by the experimenter).
2) *Process atomism.* Any process, dynamical or not, of any physical system must be viewed as an *aggregate* of isomorphic elementary operations of finite duration $\chi$, provisionally called *chronons*.
3) *Algebraic simplicity.* The dynamical and the symmetry groups (and, consequently, the operator algebra) of any physical system must be *simple* (in algebraic sense).
4) *Clifford statistics.* Chronons obey Clifford statistics.

The operator algebra of a chronon aggregate is a real Clifford algebra of a very large number of dimensions. The chronons themselves are represented by the generators (spins) $\gamma_n$ of that Clifford algebra, with the property $\gamma_n^2 = \pm 1$.

Using the algebra of an ensemble of many chronons I algebraically simplify the non-semisimple Dirac–Heisenberg algebra of relativistic quantum mechanics, unifying the space–time, energy–momentum, and spin variables of the electron. I also propose a new dynamics that reduces to Dirac's dynamics for fermions with the usual Heisenberg commutation relations in the continuum limit, when the number $N$ of elementary processes becomes infinite and their duration $\chi$ goes to zero. The complex imaginary unit $i$, now a dynamical variable, the mass term $m$, and a new spin–orbit coupling not present in the Standard Model, all appear naturally within the new simplified theory.

## 2. ELEMENTARY OPERATIONS AND CLIFFORD STATISTICS

> *. . . The situation, however, is somewhat as follows. In order to give physical significance to the concept of time, processes of some kind are required which enable relations to be established between different places.*
>
> Albert Einstein, 1955

### 2.1. Process Atomism

We start our work with the idea that *any process, dynamical or not, of any physical system should be viewed as an aggregate of isomorphic elementary operations*. We call such fundamental operations *chronons*.

The dynamics of a physical system is usually specified either by a Hamiltonian, or by a Lagrangian. It represents the time evolution of the system under study between our initial and final determination actions. The simplest and more or less representative example of a dynamical process is that of time evolution of an electron subjected to an external electromagnetic field.

Let us assume that a system of classical charges, magnets, and current carrying coils and wires is distributed in a definite way throughout the experimental region. It produces some definite electric and magnetic fields, in which the electron under study moves. The action of the field on the electron defines electron's dynamics.

We can change the dynamics by rearranging the elements of the field-producing system—the charges, the wires, and so on. If the dynamics is composed of elementary operations, as is assumed by our atomistic hypothesis, the change in the dynamics must be accomplished by permutation of the underlying chronons.

If we want to have different dynamics, the permutation has to have a definite effect on the aggregate of our fundamental operations. In other words, we need to assign a statistics to the chronons, and that statistics must be *nonabelian*.[2]

Thus, the first question that has to be answered in setting up an algebraic quantum theory of composite processes is: What statistics do the elementary actions have?

Ordinarily, processes (space–time event, field values, etc.) have been assumed, though implicitly, to be distinguishable, being addressed by their space–time coordinates. If considered classical, they obey Maxwell–Boltzmann statistics.

In our theory, the elementary processes of nature obey Clifford statistics, which is similar to be the spinorial statistics of Nayak and Wilczek. We apply this statistics first to toy models of particles in ordinary space–time simply to familiarize ourselves with its properties. In our construction the representation space of the permutation group is the whole (spinor) space of the composite. The permutation group is not assumed to be a symmetry of the Hamiltonian or of its ground subspace. It is used not as a symmetry group but as a dynamical group of an aggregate.

## 2.2. Quantification Procedures and Statistics

Apart from the atomism of actions discussed above, we suggest that *at higher energies the present complex quantum theory with its unitary group will expand into a real quantum theory with an orthogonal group, broken by an approximate i operator at lower energies*. To implement this possibility and to account for double-valuedness of spin in Nature, we develop a new real quantum double-valued statistics. In this statistics, called Clifford, we represent a swap (12) of two quanta by the difference $\gamma_1 - \gamma_2$ of the corresponding Clifford units. The operator algebra of an ensemble is the Clifford algebra over a one-body real Hilbert space.

[2] Recall that a statistics is *abelian* if it represents the permutation group $S_N$ on the $N$ members of an ensemble by an abelian group of operators in the $N$-body mode space.

The usual Fermi–Dirac or Bose–Einstein statistics are abelian. In a sense they are trivially abelian because they represent each permutation by a number, a projective representation of the identity operator. Entities with scalar statistics are regarded as *indistinguishable*. Thus bosons and fermions are indistinguishable.

Nonabelian statistics describe distinguishable quanta.

One nontrivial example of nonabelian statistics was given by Nayak and Wilczek (1996) and Wilczek (1998b) in their work on quantum Hall effect. It was based on the earlier work on nonabelions of Read and Moore (1992) and Moore and Read (1990). Read and Moore use a subspace corresponding to the degenerate ground mode of some realistic Hamiltonian as the representation space for a nonabelian representation of the permutation group $S_{2n}$ acting on the composite of $2n$ quasiparticles in the fractional quantum Hall effect. This statistics, Wilczek showed, represents the permutation group on a spinor space, permutations being represented by noncommuting spin operators. The quasiparticles of Read and Moore and of Wilczek and Nayak are distinguishable, but their permutations leave the ground subspace invariant.

Unlike the Maxwell–Boltzmann, Fermi–Dirac, Bose–Einstein, and parastatistics, which are tensorial and single-valued, and unlike anyons, which are confined to two dimensions, Clifford statistics is multivalued and works for any dimensionality. Interestingly enough, a similar statistics was proposed by Nayak and Wilczek (1996) for the excitations in the theory of fractional quantum Hall effect.

To develop some feel for this new statistics, we first apply it to toy quanta. We distinguish between the two possibilities: a real Clifford statistics and a complex one. A complex-Clifford example describes an ensemble with the energy spectrum of a system of spin-1/2 particles in an external magnetic field. This (maybe somewhat prematurely) supports the proposal that the double-valued rotations—spin—seen at current energies arise from double-valued permutations—swap—to be seen at higher energies. Another toy with real Clifford statistics illustrates how an effective imaginary unit $i$ can arise naturally within a real quantum theory.

### *2.2.1. Quantifiers*

All the common statistics, including Fermi–Dirac (F-D), Bose–Einstein (B-E), and Maxwell–Boltzmann (M-B), may be regarded as various prescriptions for constructing the algebra of an ensemble of many individuals from the vector space of one individual. These prescriptions convert "yes-or-no" questions about an individual into "yes-or-no" and "how-many" questions about an ensemble of similar individuals. Sometimes these procedures are called "second quantization." This terminology is unfortunate for obvious reasons. We will not use it in our work. Instead, we will speak of *quantification*.

We use the operational formulation of quantum theory presented in Appendix A. As we pointed out, kets represent initial modes (of preparation), bras represent final modes (of registration), and operators represent intermediate operations on quantum. The same applies to an ensemble of several quanta.

Each of the *usual* statistics may be defined by an associated linear mapping $Q^\dagger$ that maps any one-body initial mode $\psi$ into a many-body creation operator:

$$Q^\dagger : V_{\mathrm{I}} \to \mathcal{A}_{\mathrm{S}}, \qquad \psi \mapsto Q^\dagger \psi =: \hat{\psi}. \tag{1}$$

Here $V_{\mathrm{I}}$ is the initial-mode vector space of the individual I and $\mathcal{A}_S = \mathrm{End}\, V_{\mathrm{S}}$ is the operator (or endomorphism) algebra of the quantified system S. The $\dagger$ in $Q^\dagger$ reminds us that $Q^\dagger$ is contragredient to the initial modes $\psi$. We write the mapping $Q^\dagger$ to the left of its argument $\psi^\dagger$ to respect the conventional Dirac order of cogredient and contragredient vectors in a contraction.

Dually, the final modes $\psi^\dagger$ of the dual space $V_{\mathrm{I}}^\dagger$ are mapped to annihilators in $\mathcal{A}_{\mathrm{S}}$ by the linear operator $Q$,

$$Q : V_{\mathrm{I}}^\dagger \to \mathcal{A}_{\mathrm{S}}, \qquad \psi^\dagger \mapsto \psi^\dagger Q =: \hat{\psi}^\dagger. \tag{2}$$

We call the transformation $Q$ the *quantifier* for the statistics. $Q$ and $Q^\dagger$ are tensors of the type

$$Q = \left(Q^{aB}{}_C\right), \qquad Q^\dagger = \left(Q^\dagger{}_a{}^C{}_B\right), \tag{3}$$

where $a$ indexes a basis in the one-body space $V_\mathrm{I}$ and $B, C$ index a basis in the many-body space $V_\mathrm{S}$.

The basic creators and annihilators associated with an arbitrary basis $\{e_a \mid a = 1, \ldots, N\} \subset V_\mathrm{I}$ and its reciprocal basis $\{e^a \mid a = 1, \ldots, N\} \subset V_\mathrm{I}^\dagger$ are then

$$Q^\dagger e_a := \hat{e}_a =: Q^\dagger{}_a, \tag{4}$$

and

$$e^a Q := \hat{e}^a =: Q^a. \tag{5}$$

The creator and annihilator for a general initial mode $\psi$ are

$$\begin{aligned} Q^\dagger \left(e_a \psi^a\right) &= Q^\dagger{}_a \psi^a, \\ \left(\phi^\dagger{}_a e^a\right) Q &= \phi^\dagger{}_a Q^a, \end{aligned} \tag{6}$$

respectively.

We require that quantification respects the adjpoint †. This relates the two tensors $Q$ and $Q^\dagger$:

$$\psi^\dagger Q = (Q^\dagger \psi)^\dagger. \tag{7}$$

The rightmost † is the adjoint operation for the quantified system. Therefore

$$\hat{e}_a^\dagger = M_{ab} \hat{e}^b, \tag{8}$$

with $M_{ab}$ being the metric, the matrix of the adjoint operation, for the individual system.

A few remarks on the use of creation and annihilation operators would now be appropriate.

If we choose to work exclusively with an $N$-body system, then all the initial and final selective actions (projections, or yes–no experiments) on that system can be taken as *simultaneous* sharp production and registration of *all* the $N$ particles in the composite with no need for one-body creators and annihilators. The theory would resemble that of just one particle. The elementary nonrelativistic quantum theory of atom provides such an example.

In real experiments much more complicated processes occur. The number of particles in the composite may vary, and if a special *vacuum* mode is introduced, then those processes can conveniently be described by postulating elementary operations of one-body creation and annihilation. Using just the notion of the vacuum mode and a simple rule by which the creation operators act on the many-body modes, it is possible to show (Weinberg, 1995) that *any* operator of *such* a many-body theory may be expressed as a sum of products of creation and annihilation operators.

In physics shifts in description are very frequent, especially in the theory of solids. The standard example is the phonos description of collective excitations in crystal lattice. There the fundamental system is an ensemble of a fixed number of ions without any special vacuum mode. An equivalent description is in terms of a variable number of phonons, their creation and annihilation operators, and the vacuum.

It is thus possible that a deeper theory underlying the usual physics might be based on a completely new kind of description. Finkelstein some time ago (personal conversations with Finkelstein, 1999) suggested that the role of atomic processes in such a theory might be played by swaps (or permutations) of quantum space–time events. Elementary particles then would be the excitations of a more fundamental system. The most natural choice for the swaps is provided by the differences of Clifford units (F8) defined in Appendix F.

All this prompts us to generalize from the common statistics to more general statistics.

A *linear statistics* will be defined by a linear correspondence $Q^\dagger$ called the quantifier,

$$Q^\dagger : V_{\mathrm{I}} \to \mathcal{A}_{\mathrm{S}}, \qquad \psi \mapsto Q^\dagger \psi =: \hat{\psi}, \tag{9}$$

[compare (1)] from one-body modes to many-body operators, †-algebraically generating the algebra $\mathcal{A}_{\mathrm{S}} := \mathrm{End}\ V_{\mathrm{S}}$ of the many-body theory.

*In general $Q^\dagger$ does not produce a creator and $Q$ does not produce an annihilator, as they do in the common statistics.*

### *2.2.2. Constructing the Quantified Algebra*

We construct the quantified algebra $\mathcal{A}_{\mathrm{S}}$ from the individual space $V_{\mathrm{I}}$ in four steps (personal conversations with Finkelstein, 1999) as follows (see Appendix C for details.):

1. We form the quantum algebra $\mathcal{A}(V_{\mathrm{I}})$, defined as the free † algebra generated by (the vectors of) $V_{\mathrm{I}}$. Its elements are all possible iterated sums and products and †-adjoints of the vectors of $V_{\mathrm{I}}$. We require that the operations $(+, \times, \dagger)$ of $\mathcal{A}(V_{\mathrm{I}})$ agree with those of $V_{\mathrm{I}}$ where both are meaningful.
2. We construct the ideal $\mathcal{R} \subset \mathcal{A}$ of all elements of $\mathcal{A}(V_{\mathrm{I}})$ that vanish in virtue of the statistics. It is convenient and customary to define $\mathcal{R}$ by a set of expressions $\mathbf{R}$, such that the commutation relations between elements of $\mathcal{A}(V_{\mathrm{I}})$ have the form $r = 0$ with $r \in \mathbf{R}$. Then $\mathcal{R}$ consists of all elements of

$\mathcal{A}(V_{\rm I})$ that vanish in virtue of the commutation relations and the postulates of a †-algebra.

Let **R** be closed under †. Let $\mathcal{R}_0$ be the set of all evaluations of all the expressions in **R** when the variable vectors $\psi$ in these expressions assume any values $\psi \in V_{\rm I}$. Then $\mathcal{R} = \mathcal{A}(V_{\rm I})\mathcal{R}_0\mathcal{A}(V_{\rm I})$. Clearly, $\mathcal{R}$ is a *two-sided* ideal of $\mathcal{A}(V_{\rm I})$.

3. We form the quotient algebra

$$\mathcal{A}_{\rm S} = \mathcal{A}(V_{\rm I})/\mathcal{R}, \tag{10}$$

by identifying elements of $\mathcal{A}(V_{\rm I})$ whose differences belong to $\mathcal{R}$.

If one is interested in the system with a fixed number of particles, one adds a step:

4. Take the subalgebra $PA_{\rm S}P$, where $P$ is the projection on the selected eigenspace of the *number operator* $N_{\rm S}$,

$$N_{\rm S} := \sum_{a=1}^{N} \hat{e}_a \hat{e}^a, \tag{11}$$

where $k$ labels the basis elements of $V_{\rm I}$ and $N = \dim V_{\rm I}$.

Thus in all the usual statistics, $\mathcal{A}_{\rm S}$ is the sum

$$\mathcal{A}_{\rm S} = \mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2 + \cdots \tag{12}$$

of 0-, 1-, 2-, ... particle algebras. In Clifford statistics (see below), on the other hand, the "number operator" is

$$N_{\rm C} = \sum_{a,b=1}^{N} g_{ab}\gamma^a\gamma^b = N, \tag{13}$$

($\gamma^a$ being the Clifford generators), which means that the number $N$ of elements in the Clifford ensemble is fixed by the dimensionality of the corresponding one-quantum Hilbert space.

In any case, as the result of the above construction, $Q^\dagger$ maps each vector $\psi \in V_{\rm I}$ into its residue class $\psi + \mathcal{R}$.

Historically, physicists carried out one special quantification first. Since in classical physics one multiplies phase spaces when quantifying, they assumed that in quantum mechanics one should multiply Hilbert spaces, forming the tensor product

$$V_{\rm S} = \bigotimes_{p=0}^{N} V_{\rm I} = V_{\rm I}^N \tag{14}$$

of $N$ individual spaces $V_{\rm I}$. To improve agreement with experiment, they removed degrees of freedom in the tensor product connected with permutations, reducing

$V_I^N$ to a subspace $P_S V_I^N$ invariant under all permutations of individuals. Here $P_S$ is a projection operator characterizing the statistics. The many-body algebra was then taken to be the algebra of linear operators on the reduced space: $\mathcal{A}_S = \text{End } P_S V_I^N$.

We call a statistics built in that way on a subspace of the tensor algebra over the one-body initial mode space, a *tensorial* statistics. Tensorial statistics represents permutations in a single-valued way. The common statistics are tensorial.

Linear statistics is more general than tensorial statistics, in that the quotient algebra $\mathcal{A} = \mathcal{A}(V_I)/\mathcal{R}$ defining a linear statistics need not be the operator algebra of any subspace of the tensor space Ten $V_I$ and need not be single-valued. Commutation relations permit more general statistics than projection operators do. For example, anyone statistics is linear but not tensorial.

For another example, $\mathcal{A}_S$ may be the endomorphism algebra of a spinor space constructed from the quadratic space $V_I$. Such a statistics we call a *spinorial statistics*. Clifford statistics, the main topic of this work, is a spinorial statistics. Linear statistics includes both spinorial and tensorial statistics (personal conversations with Finkelstein, 1999).

The F-D, B-E, and M-B statistics are readily presented as tensorial statistics. We give their quantifiers next (Finkelstein, 1996). We then generalize to spinorial, nontensorial, statistics.

### *2.2.3. Standard Statistics*

*Maxwell–Boltzmann statistics*. A classical M-B aggregate is a sequence (up to isomorphism) and $Q = \text{Seq}$, the *sequence*-forming quantifier. The quantum individual I has a Hilbert space $V = V_I$ over the field $\mathbb{C}$. The vector space for the quantum sequence then is the (contravariant) tensor algebra $V_S = \text{Ten } V_I$, whose product is the tensor product $\otimes$

$$V_S = \text{Tan } V_I \tag{15}$$

with the natural induced †. The kinematic algebra $\mathcal{A}_S$ of the sequence is the †-algebra of endomorphisms of Ten $V_I$, and is generated by $\psi \in V_I$ subject to the generating relations

$$\hat{\psi}^\dagger \hat{\phi} = \psi^\dagger \phi. \tag{16}$$

The left-hand side is an operator product, and the right-hand side is the contraction of the dual vector $\psi^\dagger$ with the vector $\phi$, with an implicit unit element $1 \in \mathcal{A}_S$ as a factor.

*Fermi–Dirac statistics*. Here $Q = \text{Set}$, the *set*-forming quantifier. The kinematic algebra for the quantum set has defining relations

$$\hat{\psi}\hat{\phi} + \hat{\phi}\hat{\psi} = 0, \qquad \hat{\psi}^\dagger \hat{\phi} + \hat{\phi}\hat{\psi}^\dagger = \hat{\psi}^\dagger \phi, \tag{17}$$

for all $\psi, \phi \in V_I$.

*Bose–Einstein statistics*. Here $Q = \text{Sib}$, the *sib*-forming quantifier. The sib-generating relations are

$$\hat{\psi}\hat{\phi} - \hat{\phi}\hat{\psi} = 0, \qquad \hat{\psi}^{\dagger}\hat{\phi} - \hat{\phi}\hat{\psi}^{\dagger} = \hat{\psi}^{\dagger}\phi, \tag{18}$$

for all $\psi, \phi \in V_{\mathrm{I}}$.

The individuals in each of the discussed quantifications, by construction, have the same (isomorphic) initial spaces. We call such individuals *isomorphic*.

### *2.2.4. The Representation Principle*

If we have defined how, for example, one translates individuals, this should define a way to translate the ensemble. We thus impose the following.

*Requirement of a quantification. Any unitary transformation on an individual quantum entity induces a unitary transformation on the quantified system, defined by the quantifier.*

This does not imply that, for example, the actual time-translation of an ensemble is carried out by translating the individuals: That would mean that the Hamiltonians combine additively, without interaction. The representation principle states only that there is a well-defined time-translation without interaction. This gives a physical meaning to interaction: it is the difference between the induced time-translation generator and the actual one.

Thus we posit that any †-unitary transformation

$$U : V_{\mathrm{I}} \to V_{\mathrm{I}}, \qquad \psi \mapsto U\psi$$

of the individual ket-space $V_{\mathrm{I}}$, also act naturally on the quantified mode space $V_{\mathrm{S}}$ through an operator

$$\hat{U} : V_{\mathrm{S}} \to V_{\mathrm{S}}, \qquad \psi \mapsto \hat{U}\psi,$$

and on the algebra $\mathcal{A}_{\mathrm{S}}$ according to

$$\hat{U} : \mathcal{A}_{\mathrm{S}} \to \mathcal{A}_{\mathrm{S}}, \qquad \hat{\psi} \mapsto \widehat{U\psi} = \hat{U}\hat{\psi}\hat{U}^{-1}. \tag{19}$$

This is the *representation principle*.

*Recall*. Infinitesimal form of $U : V_{\mathrm{I}} \to V_{\mathrm{I}}$ is

$$U = 1 + G\,\delta\theta, \tag{20}$$

where $G = -G^{\dagger} : V_{\mathrm{I}} \to V_{\mathrm{I}}$ is the anti-Hermitian generator and $\delta\theta$ is an infinitesimal parameter. The infinitesimal anti-Hermitian generators $G$ make up the Lie algebra $d\mathrm{U}_{\mathrm{I}}$ of the unitary group $\mathrm{U}_{\mathrm{I}}$ of the one-body theory.

In terms of generators the representation principle is expressed by

$$\hat{G} : \hat{\psi} \mapsto \widehat{G\psi} = [\hat{G}, \hat{\psi}]. \tag{21}$$

*2.2.5. The Quantified Generators*

Since

$$G = \sum_{a,b} e_a G^a{}_b e^b \tag{22}$$

holds by the completeness of the dual bases $e_a$ and $e^a$, we express the quantified generator $\hat{G}$ by[3]

$$\hat{G} := Q^\dagger G Q = \sum_{a,b} Q^\dagger{}_a G^a{}_b Q^b \equiv \sum_{a,b} \hat{e}_a G^a{}_b \hat{e}^b. \tag{23}$$

*2.2.6. Checking for the Usual Statistics*

The representation principle holds for the usual F-D and B-E statistics. It will also hold for the Clifford statistics, as we show below.

**Proposition.** *If Q is a quantifier for F-D or B-E statistics then*

$$[\hat{G}, Q^\dagger \psi] = G Q^\dagger \psi \tag{24}$$

hold for all anti-Hermitian generators $G$.

**Proof:** We have

$$\begin{aligned}
[\hat{G}, Q^\dagger \psi] &= G^a{}_b \left(\hat{e}_a \hat{e}^b Q^\dagger \psi - Q^\dagger \psi \hat{e}_a \hat{e}^b\right) \\
&= G^a{}_b \left(\hat{e}_a (e^b \psi + (-1)^\kappa Q^\dagger \psi \hat{e}^b) - Q^\dagger \psi \hat{e}_a \hat{e}^b\right) \\
&= G^a{}_b \hat{e}_a e^b \psi \\
&= G^a{}_b \hat{e}_a \psi^b \\
&= G Q^\dagger \psi.
\end{aligned} \tag{25}$$

Here $\kappa = 1$ for Fermi statistics and 0 for Bose.

*2.2.7. Quantification and Commutator Algebras*

If $\mathcal{A}$ is any algebra, by the commutator algebra $\Delta\mathcal{A}$ of $\mathcal{A}$ we mean the Lie algebra on the elements of $\mathcal{A}$ whose product is the commutator $[a, b] = ab - ba$ in $\mathcal{A}$. By the commutator algebra of a quantum system **I** we mean that of its operator algebra $\mathcal{A}_{\mathrm{I}}$.

In the usual cases of Bose and Fermi statistics (and not in the cases of complex and real Clifford statistics discussed below!) the quantification rule (23) defines a

[3] According to Weinberg (1995), this rule for quantifying *observables* was first given by Heisenberg and Pauli (1929, 1930) in their early work on quantum field theory.

Lie isomorphism, $\Delta\mathcal{A}_{\mathrm{I}} \to \Delta\mathcal{A}_{\mathrm{S}}$, from the commutator algebra of the individual to that of the quantified system.

**Proposition.** *For two (arbitrary) operators $H$ and $P$ acting on $V_{\mathrm{I}}$,*

$$\widehat{[H, P]} = [\hat{H}, \hat{P}]. \tag{26}$$

**Proof:**

$$\begin{aligned}
[\hat{H}, \hat{P}] &= \hat{H}\hat{P} - \hat{P}\hat{H} \\
&= \hat{e}_r H^r{}_s \hat{e}^s \hat{e}_t P^t{}_u \hat{e}^u - \hat{e}_t P^t{}_u \hat{e}^u \hat{e}_r H^r{}_s \hat{e}^s \\
&= H^r{}_s P^t{}_u \left(\hat{e}_r \hat{e}^s \hat{e}_t \hat{e}^u - \hat{e}_t \hat{e}^u \hat{e}_r \hat{e}^s\right) \\
&= H^r{}_s P^t{}_u \left(\hat{e}_r \left(\delta^s_t \pm \hat{e}_t \hat{e}^s\right) \hat{e}^u - \hat{e}_t \hat{e}^u \hat{e}_r \hat{e}^s\right) \\
&= H^r{}_s P^t{}_u \left(\hat{e}_r \delta^s_t \hat{e}^u \pm \hat{e}_r \hat{e}_t \hat{e}^s \hat{e}^u - \hat{e}_t \hat{e}^u \hat{e}_r \hat{e}^s\right) \\
&= H^r{}_s P^t{}_u \left(\hat{e}_r \delta^s_t \hat{e}^u \pm \hat{e}_t \hat{e}_r \hat{e}^u \hat{e}^s - \hat{e}_t \hat{e}^u \hat{e}_r \hat{e}^s\right) \\
&= H^r{}_s P^t{}_u \left(\hat{e}_r \delta^s_t \hat{e}^u \pm \hat{e}_t \left(\mp \delta^u_r \pm \hat{e}^u \hat{e}_r\right) \hat{e}^s - \hat{e}_t \hat{e}^u \hat{e}_r \hat{e}^s\right) \\
&= H^r{}_s P^t{}_u \left(\hat{e}_r \delta^s_t \hat{e}^u - \hat{e}_t \delta^u_r \hat{e}^s\right) \\
&= \hat{e}_r \left(H^r{}_t P^t{}_u - P^t{}_u H^r{}_s\right) \hat{e}^u \\
&= \widehat{[H, P]}.
\end{aligned} \tag{27}$$

This implies that for B-E and F-D statistics, the quantification rule (23) can be extended from the unitary operators and their anti-Hermitian generators to the whole operator algebra (including observables) of the quantified system.

## 2.3. Clifford Statistics

### *2.3.1. Clifford Quantification*

Now let the one-body mode space $V_{\mathrm{I}} = \mathbb{R}^{N_+, N_-} = N_+\mathbb{R} \oplus N_-\mathbb{R}$ be a real quadratic space of dimension $N = N_+ + N_-$ and signature $N_+ - N_-$. Denote the symmetric metric form of $V_{\mathrm{I}}$ by $g = (g_{ab}) := (e_a^\dagger e_b)$. We do not assume that $g$ is positive-definite.

Clifford statistics (9) is defined by

(1) the Clifford-like generating relations

$$\hat{\psi}\hat{\phi} + \hat{\phi}\hat{\psi} = \frac{\zeta}{2}\psi^\dagger\phi \tag{28}$$

for all $\phi, \psi \in V_{\mathrm{I}}$, where $\zeta$ is a $\pm$ sign that can have either value;

(2) the Hermiticity condition (7)

$$\hat{e}_a^\dagger = g_{ab}\hat{e}^b; \tag{29}$$

(3) a rule for raising and lowering indices

$$\hat{e}_a := \zeta' g_{ab}\hat{e}^b, \tag{30}$$

where $\zeta'$ is another $\pm$ sign, and

(4) the rule (23) to quantify one-body generators.

Here $\zeta = \pm 1$ covers the two different conventions used in the literature. Later we will see that $\zeta = \zeta'$ and that $\zeta = \zeta' = +1$ and $\zeta = \zeta' = -1$ are both allowed physically at the present theoretical stage of development. They lead to two different real quantifications, with either Hermitian or anti-Hermitian Clifford units.

For the quantified basis, elements of $V_{\mathrm{I}}$ (28) leads to

$$\hat{e}_a\hat{e}_b + \hat{e}_b\hat{e}_a = \frac{\zeta}{2} g_{ab}. \tag{31}$$

The $\psi$s, which are assigned grade 1 and taken to be either Hermitian or anti-Hermitian, generate a graded †-algebra that we call the *free Clifford †-algebra* associated with $\mathbb{R}^{N_+,N_-}$ and write as $\mathrm{Cliff}(N_+, N_-) \equiv \mathrm{Cliff}(N_\pm)$.

In assuming a real vector space of quantum modes instead of a complex one, we give up $i$-invariance but retain quantum superposition $a\psi + b\psi$ with real coefficients. Our theory is nonlinear from the complex point of view.[4]

### 2.3.2. *Cliffordons and Their Permutations*

Clifford statistics assembles its quanta, *cliffordons*, individually described by vectors into a composite described by spinors, which we call a *squadron*.

A cliffordon is a hypothetical quantum-physical entity, like an electron, not to be confused with a mathematical object like a spinor or an operator. We cannot describe a cliffordon completely, but we represent our actions on a squadron of cliffordons adequately by operators in a Clifford algebra of operators. One encounters cliffordons only in permuting them, never in creating or annihilating them as individuals.

Clifford statistics, unlike the more familiar particle statistics (Berezin, 1966; Feynman, 1972; Negele and Orland, 1988), provides no creators or annihilators. With each individual mode $e_a$ of the quantified system they associate a Clifford unit $\gamma_a = 2Q_a^\dagger$.

[4] Others considered nonlinear quantum theories, but gave up real superposition as well as $i$-invariance (Weinberg, 1989).

In the standard statistics there is a natural way to represent permutations of individuals in the $N$-body composite. Each $N$-body ket is constructed by successive action of $N$ creation operators on the special vacuum mode. Any permutation of individuals can be achieved by permuting these creation operators in the product. The identity and alternative representations of the permutation group $S_N$ in the B-E and F-D cases then follow from the defining relations of the corresponding statistics.

In the case of Clifford statistics, some things are different. There is still an operator associated with each cliffordon; now it is a Clifford unit. Permutations of cliffordons are still represented by operators on a many-body † space. But the mode space on which these operators act is now a spinor space, and its basis vectors are not constructed by creation operators acting on a special "vacuum" ket.

We may represent any swap (transposition of two cliffordons say 1 and 2) by the difference of the corresponding Clifford units

$$t_{12} := \frac{1}{\sqrt{2}}(\gamma_1 - \gamma_2). \tag{32}$$

and represent an arbitrary permutation, which is a product of elementary swaps, by the product of their representations. That is, as direct computation shows, this defines a projective homomorphism from $S_N$ into the Clifford algebra generated by the $\gamma_k$. For details, see Appendix F.

By definition, the number $N$ of cliffordons in a squadron is the dimensionality of the individual initial mode space $V_{\mathrm{I}}$. $N$ is conserved rather trivially, commuting with every Clifford element. We can change this number only by varying the dimensionality of the one-body space. In one use of the theory, we can do this, for example, by changing the space–time 4-volume of the corresponding experimental region. Because our theory does not use creation and annihilation operators, an initial action on the squadron represented by a spinor $\xi$ should be viewed as some kind of spontaneous transition condensation into a coherent mode, analogous to the transition from the superconducting to the many-vertex mode in a type-II superconductor. The initial mode of a set or sib of (F-D or B-E) quanta can be regarded as a result of possibly entangled creation operations. That of a squadron of cliffordons cannot.

### *2.3.3. Representation Principle*

As with (25), let us verify that definition (23) is consistent in the Clifford case:

$$\begin{aligned}[\hat{G},\, Q^\dagger\psi] &= G^a{}_b \left(\hat{e}_a\hat{e}^b Q^\dagger\psi - Q^\dagger\psi\hat{e}_a\hat{e}^b\right)\\ &= \frac{1}{2}G^a{}_b \left(\hat{e}_a\psi^b + \psi_a\hat{e}^b\right)\\ &= G Q^\dagger\psi. \end{aligned} \tag{33}$$

This shows that $Q^\dagger\psi$ transforms correctly under the infinitesimal unitary transformation of $\mathbb{R}^{N_+,N_-}$ (*cf.* Dirac, 1974).

### *2.3.4. Clifford Statistics and Commutator Algebras*

In the usual statistics, the quantifier $Q$ is extended from anti-Hermitian operators to all operators. This is not the case for Clifford quantification. There the quantification of any symmetric operator is a scalar, in virtue of Clifford's law. A straightforward calculation shows that

$$\begin{aligned}[\hat{H}, \hat{P}] &= \hat{H}\hat{P} - \hat{P}\hat{H} \\ &= \zeta\zeta'\left(\frac{1}{2}[\widehat{H, P}] + \frac{1}{4}\left([\widehat{P, H^\dagger}] + [\widehat{P^\dagger, H}]\right)\right). \qquad (34)\end{aligned}$$

The three simplest cases are:

1. $H = H^\dagger, H' = H'^\dagger \Longrightarrow [\hat{H}, \hat{H}'] = 0;$
2. $H = H^\dagger, G_1 = -G_1^\dagger \Longrightarrow [\hat{H}, \hat{G}_1] = 0;$
3. $G = -G^\dagger, G' = -G'^\dagger \Longrightarrow [\hat{G}, \hat{G}'] = \zeta\zeta'[\widehat{G, G'}].$

Thus Clifford quantification respects the commutation relations for anti-Hermitian generators if and only if $\zeta = \zeta' = +1$ or $\zeta = \zeta' = -1$, but not for Hermitian observables, contrary to the Bose and Fermi quantifications, which respect both.

### *2.3.5. Complex Clifford Statistics*

The *complex* graded algebra generated by the $\psi$s with the relations (28) is called the *complex Clifford algebra* $\mathrm{Cliff}_{\mathbb{C}}(N)$ over $\mathbb{R}^{N_+,N_-}$. It is isomorphic to the full complex matrix algebra $\mathbb{C}(2^n) \otimes \mathbb{C}(2^n)$ for even $N = 2n$, and to the direct sum $\mathbb{C}(2^n) \otimes \mathbb{C}(2^n) \oplus \mathbb{C}(2^n) \otimes \mathbb{C}(2^n)$ for odd $N = 2n + 1$. We regard $\mathrm{Cliff}_{\mathbb{C}}(N)$ as the kinematic algebra of the complex Clifford composite. As a vector space, it has dimension $2^N$.[5]

For dimension $N = 3$ spinors of $\mathrm{Cliff}_{\mathbb{C}}(3)$ have as many parameters as vectors, but for higher $N$ the number of components of the spinors associated with $\mathrm{Cliff}(N_\pm)$

[5] Schur (1911) used complex spinors and complex Clifford algebra to represent permutations some years before Cartan used them to represent rotations. There is a fairly widespread view that spinors may be more fundamental than vectors, since vectors may be expressed as bilinear combinations of spinors. One of us took this direction in much of his work. Clifford statistics supports the opposite view. There a vector describes an individual, a spinor an aggregate. Wilczek and Zee (1982) seem to have been the first to recognize that spinors represent composites in a physical context, although this is implicit in the Chevalley construction of spinors within a Grassmann algebra (personal conversations with Finkelstein, 1999).

grows exponentially with $N$. The physical relevance of this irreducible double-valued (or projective) representation of the permutation group $S_N$ was recognized by Nayak and Wilczek (1996) and Wilczek (1998b) in a theory of the fractional quantum Hall effect. We call the statistics based on $\mathrm{Cliff}_{\mathbb{C}}(N)$ *complex Clifford* statistics.

### *2.3.6. Breaking the i Invariance*

Thus we cannot construct useful Hermitian variables of a squadron by applying the quantifier to the Hermitian variables of the individual cliffordon.

This is closely related to fact that the real initial mode space $\mathbb{R}^{N_\pm}$ of a cliffordon has no special operator to replace the imaginary unit $i$ of the standard complex quantum theory. The fundamental task of the imaginary element $i$ in the algebra of complex quantum physics is precisely to relate conserved Hermitian observables $H$ and anti-Hermitian generators $G$ by

$$H = -i\hbar G. \tag{35}$$

To perform this function exactly, the operator $i$ must commute exactly with all observables.

The central operators $x$ and $p$ of classical mechanics are contractions of noncentral operators $\breve{x}$ and $\breve{p} = -i\hbar\partial/\partial\breve{x}$ (Baugh *et al.*, 2001). In the limit of large numbers of individuals organized coherently into suitable condensate modes, the expanded operators of the quantum theory contract into the central operators of the classical theory. Condensations produce nearly commutative variables.

Likewise we expect the central operator $i$ to be a contraction of a noncentral operator $\breve{i}$ similarly resulting from a condensation in a limit of large numbers. In the simple expanded theory, $\breve{i}$, the correspondent of $i$, is not central.[6]

[6] One clue to the nature of $\breve{i}$ and the locus of its condensation is how the operator $i$ behaves when we combine separate systems. Since infinitesimal generators $G, G', \ldots$ combine by addition, the imaginaries $i, i', \ldots$ of different individuals must combine by identification

$$i = i' = \ldots \tag{36}$$

for (35) to hold exactly, and nearly so for (35) to hold nearly. The only other variables in present physics that combine by identification in this way are the time $t$ of classical mechanics and the space–time coordinates $x^\mu$ of field theories. All systems in an ensemble must have about the same $i$, just as all particles have about the same $t$ in the usual instant-based formulation, and all fields have about the same space–time variables $x^\mu$ in field theory. We identify the variables $t$ and $x^\mu$ for different systems because they are set by the experimenter, not the system. This suggests that the experimenter, or more generally the environment of the system, mainly defines the operator $i$. The central operators $x$, $p$ characterize a small system that results from the condensation of many particles. The central operator $i$ must result from a condensation in the environment; we take this to be the same condensation that forms the vacuum and the spatiotemporal structure represented by the variables $x^\mu$ of the standard model (personal conversations with Finkelstein, 1999).

The existence of this contracted $i$ ensures that at least approximately, every Lie commutation relation between dimensionless anti-Hermitian generators $A$, $B$, $C$ of the standard complex quantum theory,

$$[A, B] = C, \tag{37}$$

corresponds to a commutation relation between Hermitian variables $-i\hbar A$, $-i\hbar B$, $-i\hbar C$:

$$[-i\hbar A, -i\hbar B] = -i\hbar(-i\hbar C). \tag{38}$$

It also tells us that this correspondence is not exact in nature.

Stückelberg (1960) reformulated complex quantum mechanics in the real Hilbert space $\mathbb{R}^{2N}$ of twice as many dimensions by assuming a special real antisymmetric operator $J : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ commuting with all the variables of the system.

A real † or Hilbert space has no such operator. For example, in $\mathbb{R}^2$ the operator

$$E := \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix} \tag{39}$$

is a symmetric operator with an obvious spectral decomposition representing, according to the usual interpretation, two selection operations performed on the system, and cannot be written in the form $G = -J\hbar E$ relating it to some antisymmetric generator $G$ for any real antisymmetric $J$ commuting with $E$.

On the other hand, if we restrict ourselves to observable operators of the form

$$E' := \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix}, \tag{40}$$

we can use the operator $J$,

$$J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{41}$$

to restore the usual connection between symmetry transformations and corresponding observables. This restriction can be generalized to any even number of dimensions (Stückelberg, 1960).

### *2.3.7. Breakdown of the Expectation Value Formula*

For a system described in terms of a general real Hilbert space there is no simple relation of the form $G = \frac{i}{\hbar}H$ between the symmetry generators and the observables: the usual notions of Hamiltonian and momentum are meaningless in that case. This amplifies our earlier observation that Clifford quantification $A \to \hat{A}$ respects the Lie commutation relations among anti-Hermitian generators, not Hermitian observables.

Operationally, this means that selective acts of individual and quantified cliffordons use essentially different sets of filters. This is not the case for complex quantum mechanics and the usual statistics. There some important filters for the composite are simply assemblies of filters for the individuals.

Again, in the complex case the expectation value formula for an assembly

$$\mathrm{Av}X = \psi^{\dagger} X \psi / \psi^{\dagger} \psi \tag{42}$$

is a consequence of the eigenvalue principle for individuals, rather than an independent assumption (Finkelstein 1963, 1996). The argument presented in Finkelstein (1963, 1996) assumes that the individuals over which the average is taken combine with Maxwell–Boltzmann statistics. For highly excited systems this is a good approximation even if the individuals have F-D or B-E or other tensorial statistics. It is not necessarily a good approximation for cliffordons, which have spinorial, not tensorial, statistics.

### *2.3.8. SPIN-1/2 Complex Clifford Model*

In this section we present a simplest possible model of a complex Clifford composite. The resulting many-body energy spectrum is isomorphic to that of a sequence of spin-1/2 particles in an external magnetic field.

Recall that in the usual complex quantum theory the Hamiltonian is related to the infinitesimal time-translation generator $G = -G^{\dagger}$ by $G = iH$. Quantifying $H$ gives the many-body Hamiltonian. In the framework of spinorial statistics, as discussed above, this does not work, and quantification in principle applies to the anti-Hermitian time-translation generator $G$, not to the Hermitian operator $H$. Our task now is to choose a particular generator and to study its quantified properties.

We assume an even-dimensional real initial-mode space $V_{\mathrm{I}} = \mathbb{R}^{2n}$ for the quantum individual, and consider the dynamics with the simplest nontrivial time-translation generator

$$G := \varepsilon \begin{bmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{bmatrix} \tag{43}$$

where $\varepsilon$ is a constant energy coefficient.

The quantified time-translation generator $\hat{G}$ then has the form

$$\begin{aligned}\hat{G} :&= \sum_{i,j}^{N} \hat{e}_l G^l{}_j \hat{e}^j \\ &= -\varepsilon \sum_{k=1}^{n} \left(\hat{e}_{k+n}\hat{e}^k - \hat{e}_k\hat{e}^{k+n}\right) \\ &= +\varepsilon \sum_{k=1}^{n} (\hat{e}_{k+n}\hat{e}_k - \hat{e}_k\hat{e}_{k+n}) \\ &= 2\varepsilon \sum_{k=1}^{n} \hat{e}_{k+n}\hat{e}_k \\ &\equiv \frac{1}{2}\varepsilon \sum_{k=1}^{n} \gamma_{k+n}\gamma_k. \end{aligned} \tag{44}$$

By Stone's theorem, the generator $\hat{G}$ of time translation in the spinor space of the complex Clifford composite of $N = 2n$ individuals can be factored into a Hermitian $H^{(N)}$ and an imaginary unit $i$ that commutes strongly with $H^{(N)}$:

$$\hat{G} = iH^{(N)}. \tag{45}$$

We suppose that $H^{(N)}$ corresponds to the Hamiltonian and seeks its spectrum.

We note that by (44), $\hat{G}$ is a sum of $n$ commuting anti-Hermitian algebraically independent operators $\gamma_{k+n}\gamma_k$, $k = 1, 2, \ldots, n$, $(\gamma_{k+n}\gamma_k)^\dagger = -\gamma_{k+n}\gamma_k$, $(\gamma_{k+n}\gamma_k)^2 = -1^{(N)}$.

We use the well-known $2^n \times 2^n$ complex matrix representation of the $\gamma$-matrices of the complex universal Clifford algebra associated with the real quadratic space $\mathbb{R}^{2n}$ (see Appendix F). We can simultanecusly diagonalize the $2^n \times 2^n$ matrices representing the commuting operators $\gamma_{k+n}\gamma_k$, and use their eigenvalues, $\pm i$, to find the spectrum $\lambda$ of $\hat{G}$, and consequently of $H^{(N)}$.

A simple calculation shows that the spectrum of $\hat{G}$ consists of the eigenvalues

$$\lambda_k = \frac{1}{2}\varepsilon(n - 2k)i, \quad k = 0, 1, 2, \ldots, n, \tag{46}$$

with multiplicity

$$\mu_k = C_k^n := \frac{n!}{k!(n-k)!}. \tag{47}$$

The spectrum of the Hamiltonian $H^{(N)}$ is

$$E_k = \frac{1}{2}(n - 2k)\varepsilon, \tag{48}$$

with degeneracy $\mu_k$. Thus $E_k$ ranges over the interval

$$-\frac{1}{4}N\varepsilon < E < \frac{1}{4}N\varepsilon, \tag{49}$$

in steps of $\varepsilon$, with the given degeneracies.

Thus the spectrum of the structureless $N$-body complex Clifford composite is the same as that of a system of $N$ spin-1/2 Maxwell–Boltzmann particles of magnetic moment $\mu$ in a magnetic field $\vec{H}$, with the identification

$$\frac{1}{4}\varepsilon = \mu H. \tag{50}$$

Even though we started with such a simple one-body time-translation generator as (G15), the spectrum of the resulting many-body Hamiltonian possesses some complexity, reflecting the fact that the units in the composite are distinguishable, and their swaps generate the dynamical variables of the system.

This spin-1/2 model does not tell us how to swap two Clifford units experimentally. Like the phonon model of the harmonic oscillator, the statistics of the individual quanta enters the picture only through the commutation relations among the fundamental operators of the theory.

### *2.3.9. Real Clifford Statistics*

According to the Periodic Table of the Spinors (Budinich and Trautman, 1988; Lounesto, 1997; Porteous, 1995; Snygg, 1997), the free (or universal) Clifford algebra $\mathrm{Cliff}_{\mathbb{R}}(N_+, N_-)$ is algebra-isomorphic to the endomorphism algebra of a module $\Sigma(N_+, N_-)$ over a ring $\mathcal{R}(N_+, N_-)$.

The Periodic Table of the Spinors (here $\zeta = -1$):

| $N_+$ \ $N_-$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $\mathbb{R}$ | $\mathbb{R}_2$ | $2\mathbb{R}$ | $2\mathbb{C}$ | $2\mathbb{H}$ | $2\mathbb{H}_2$ | $4\mathbb{H}$ | $8\mathbb{C}$ | $\cdots$ |
| 1 | $\mathbb{C}$ | $2\mathbb{R}$ | $2\mathbb{R}_2$ | $4\mathbb{R}$ | $4\mathbb{C}$ | $4\mathbb{H}$ | $4\mathbb{H}_2$ | $8\mathbb{H}$ | $\cdots$ |
| 2 | $\mathbb{H}$ | $\mathbb{C}_2$ | $4\mathbb{R}$ | $4\mathbb{R}_2$ | $8\mathbb{R}$ | $8\mathbb{C}$ | $8\mathbb{H}$ | $8\mathbb{H}_2$ | $\cdots$ |
| 3 | $\mathbb{H}_2$ | $2\mathbb{H}$ | $4\mathbb{C}$ | $8\mathbb{R}$ | $8\mathbb{R}_2$ | $16\mathbb{R}$ | $16\mathbb{C}$ | $16\mathbb{H}$ | $\cdots$ |
| 4 | $2\mathbb{H}$ | $2\mathbb{H}_2$ | $4\mathbb{H}$ | $8\mathbb{C}$ | $16\mathbb{R}$ | $16\mathbb{R}_2$ | $32\mathbb{R}$ | $32\mathbb{C}$ | $\cdots$ |
| 5 | $4\mathbb{C}$ | $4\mathbb{H}$ | $4\mathbb{H}_2$ | $8\mathbb{H}$ | $16\mathbb{C}$ | $32\mathbb{R}$ | $32\mathbb{R}_2$ | $64\mathbb{R}$ | $\cdots$ |
| 6 | $8\mathbb{R}$ | $8\mathbb{C}$ | $8\mathbb{H}$ | $8\mathbb{H}_2$ | $16\mathbb{H}$ | $32\mathbb{C}$ | $64\mathbb{R}$ | $64\mathbb{R}_2$ | $\cdots$ |
| 7 | $8\mathbb{R}_2$ | $16\mathbb{R}$ | $16\mathbb{C}$ | $16\mathbb{H}$ | $16\mathbb{H}_2$ | $32\mathbb{H}$ | $64\mathbb{C}$ | $128\mathbb{R}$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |

(51)

The ring of coefficients $\mathcal{R}(N_+, N_-)$ varies periodically with period 8 in each of the dimensionalities $N_+$ and $N_-$ of $V_\mathrm{I}$, and is a function of signature $N_+ - N_-$ alone.

In our application the module $\Sigma(N_+, N_-)$, the spinor space supporting $\mathrm{Cliff}_{\mathbb{R}}(N_+, N_-)$, serves as the initial mode space of a squadron of $N$ real cliffordons. $\mathcal{R}(N_\pm)$ we call the *spinor coefficient ring* for $\mathrm{Cliff}_{\mathbb{R}}(N_+, N_-)$.

### *2.3.10. Emergence of a Quantum i in the Real Clifford Statistics*

The Periodic Table of the Spinors suggests another origin for the complex $i$ of quantum theory, and one that is not approximately central but exactly central. Some Clifford algebras $\mathrm{Cliff}_{\mathbb{R}}(N_+, N_-)$ have the spinor coefficient ring $\mathbb{C}$, containing an element $i$. Multiplication by this $i$ then represents an operator in the center of the Clifford algebra, which also we designate by $i$. We may use $i$-multiplication to represent the top element $\gamma^\uparrow$ whenever $\gamma^\uparrow$ is central and has square $-1$. This $i \in \mathrm{Cliff}_{\mathbb{R}}(N_\pm)$ corresponds to the $i$ of complex quantum theory.

*Examples*:

$\mathrm{Cliff}_{\mathbb{R}}(1, 0)$ is commutative;
$\mathrm{Cliff}_{\mathbb{R}}(0, 3)$ and $\mathrm{Cliff}_{\mathbb{R}}(5, 0)$ are noncommutative.

Triads or pentads of such cliffordons could underlie the physical "elementary" particles, giving rise to complex quantum mechanics within the real.

Let us consider $\mathrm{Cliff}_{\mathbb{R}}(0, 3) = \mathbb{C}(2)$. Its Pauli representation is $\gamma_1 := i\sigma_1$, $\gamma_2 := i\sigma_2$, $\gamma_3 := i\sigma_3$ with $\zeta = -1$. Choose a particular one-cliffordon dynamics of the form

$$G := \begin{bmatrix} 0 & V & 0 \\ -V & 0 & \varepsilon \\ 0 & -\varepsilon & 0 \end{bmatrix}. \tag{52}$$

Quantification (23) of $G$ gives

$$\hat{G} = iH^{(-3)} \tag{53}$$

with the Hamiltonian

$$H^{(-3)} = \frac{1}{2}\begin{bmatrix} V & \varepsilon \\ \varepsilon & -V \end{bmatrix}. \tag{54}$$

This is also the Hamiltonian for a generic two-level quantum-mechanical system (with the energy separation $\varepsilon$) in an external potential field $V$, like the ammonia molecule in a static electric field discussed in Feynman *et al.* (1965).

## 3. ALGEBRAIC SIMPLICITY AND DIRAC'S DYNAMICS

> *. . . it is contrary to the mode of thinking in science to conceive of a thing (the space–time continuum) which acts itself, but which cannot be acted upon.*
>
> Albert Einstein, 1955

### 3.1. Algebraic Simplicity

#### *3.1.1. Simple Theories and Unification Programs*

A *simple* theory is one with simple (irreducible) dynamical and symmetry groups. What is not simple or semisimple we call *compound*. A *contraction* of a theory is a deformation of the theory in which some physical scale parameter, called the *simplifier*, approaches a singular limit, taken to be 0 with no loss of generality. The contraction of a simple theory is in general compound (Inönü, 1964; Inönü and Wigner, 1952; Segal, 1951). By *simplification* we mean the more creative, nonunique liverse process, finding a simple theory that contracts to a given compound theory and agrees better with experiment. The main revolutions in physics of the twentieth century were simplifications with simplifiers $c, G, \hbar$.

One sign of a compound theory is a breakdown of reciprocity, the principle that every coupling works both ways. The classic example is Galilean relativity. There reciprocity between space and time breaks down, boosts couple time into space, and there is no reciprocal coupling. Special relativity established reciprocity by replacing the compound Galilean bundle of space fibers over the time base by the simple Minkowski space–time. Had Galileo insisted on simplicity and reciprocity he could have formulated special relativity in the seventeenth century (unless he were to choose SO(4) instead of SO(1, 3)). Every bundle theory violates reciprocity as much as Galileo's. The bundle group couples the base to the fiber but not conversely. Every bundle theory cries out for simplification.

This now requires us to establish reciprocity between space–time (base coordinates) $x^\mu$ and energy-momenta (fiber coordinates) $p_\mu$.[7]

Einstein's gravity theory and the Standard Model of the other forces are bundle theories, with field space as fiber and space–time as base. Therefore these theories are ripe for simplification (Baugh *et al.*, 2001). Here we simplify a spinor theory, guided by criteria of experimental adequacy, operationality, causality, and finity.

[7] Segal (1951) postulated $x \leftrightarrow p$ symmetry exactly on grounds of algebraic simplicity; his work stimulated that of Inönü and Wigner, and ours. Born (1949) postulated $x \leftrightarrow p$ reciprocity, on the grounds that it is impossible in principle to measure the usual four-dimensional interval of two events within an atom. We see no law against measuring space–time coordinates and intervals at that gross scale. We use his term "reciprocity" in a broader sense that includes his.

Classical field theory is but a singular limit of quantum field theory; it suffices to simplify the quantum field theory. Quantum field theory in turn we regard as many-quantum theory. Its field variables all arise from spin variables of single quanta. For example, a spinor field arises from the theory of a single quantum of spin 1/2 by a transition to the many-body theory, or *quantification* (the transition from the one-body to the many-body theory, converting yes-or-no predicates about an individual into how-many predicates about an aggregate of isomorphic individuals; as distinct from quantization). To unify field with space–time in quantum field theory, it suffices to unify spin with space–time in the one-quantum theory, and to quantify the resulting theory. We unify in this work and quantify in a sequel.

Some unification programs concern themselves with simplifying just the internal symmetry group of the elementary particles, ignoring the fracture between the internal and external variables. They attempt to unify (say) the hypercharge, isospin, and color variables, separate from the space–time variables. Here we close the greatest wound first, expecting that the internal variables will unite with each other when they unite with the external variables as uniting space with time incidentally unified the electric and magnetic fields. We represent space–time variables $x^{\mu}$ and $p_{\mu}$ as approximate descriptions of many spin variables, in one quantum-spin – space–time structure described in a higher-dimensional spin algebra. This relativizes the split between field and space–time, as Einstein relativized the split between space and time.[8]

The resulting quantum atomistic space–time consists of many small exactly Lorentz-invariant isomorphic quantum elements which we call *chronons*.[9]

Simplifying a physical theory generally detaches us from a supporting condensate.[10] In the present situation of physics the prime condensate is the ambient vacuum. Atomizing space–time enables us to present the vacuum as a condensate of a simple system, and to detach from it in thought, by a phase transition, a space–time meltdown.

### *3.1.2. Chronon Statistics*

Chronons carry a fundamental time-unit $\chi$, one of our simplifiers. Finkelstein (see Appendix D) has argued that $\chi$ is much greater than the Planck time and is

[8] A different approach to the quantum-field – space–time unification is provided by supersymmetry. It will not be considered in this work.

[9] Feynman *et al.* attempted to atomize space or space–time into quantum spins. Feynman wrote a space–time vector as the sum of a great many Dirac spin-operator vectors (Feynman's private communication with Finkelstein, 1999), $x^{\mu} \sim \Sigma_n \gamma^{\mu}(n)$, Penrose dissected the sphere $S^2$ into a spin network (Penrose, 1971); his work inspired this program. Weizsäcker (1986), attempted a cosmology of spin-1/2 urs. The respective groups are Feynman's SO(3, 1), Penrose's SO(3), Weizsäckers SU(2), and our $SO(3N, 3N)$ $(N \gg 1)$.

[10] For Galileo and Kepler, the condensate was the Earth's crust, and to detach from it they moved in thought to a ship or the moon, respectively (Galilei, 1967; Kepler, 1967).

on the order of the Higgs time $\hbar/M_{\rm H}c^2$.[11] We now replace the classical Maxwell–Boltzmann statistics of space–time events with the simple Clifford–Wilczek statistics appropriate for distinguishable isomorphic units. This enormously reduces the problem of forming a theory.

We single out two main quantifications in field theories like gravitation and the Standard Model:

A classical quantification assembles a space–time from individual space–time points.

A separate quantification constructs a many-quantum theory or quantum field theory from a one-quantum theory on that space–time.

In standard physics the space–time quantification tacitly assumes Maxwell–Boltzmann statistics for the elements of space–time, and the field quantification uses Fermi–Dirac or Bose–Einstein statistics. The simplified theory we propose uses one Clifford quantification for all of these purposes.

In this paper we work only with one-quantum processes of $N \gg 1$ chronons. To describe several quanta and their interactions, getting closer to field theory and experiment, will require no further quantification, but only an additional internal combinatory structure that is readily accomodated within the one Clifford–Wilczek quantification.

For reader's convenience we briefly recapitulate the main points of the previous sections.

*Statistics*. One defines the statistics of an (actual, not virtual) aggregate by defining how the aggregate transforms under permutations of its units. That is, to describe $N$ units with given unit mode space $V_1$ we give, first, the mode space $V_N$ of the aggregate quantum system and, second, a simple representation $R_N : S_N \to \text{End } V_N$ of the permutation group $S_N$ on the given $N$ units by linear operators on $V_N$. This also defines the quantification that converts yes-or-no questions about the individual into how-many questions about a crowd.

In *Clifford statistics* End $V_N$ is a Clifford algebra $C = \text{Cliff}(V_1)$, and so $V_N$ is a spinor space for that Clifford algebra, with $C = \text{End } V_N$. We write $C_1$ for the first-degree subspace of $C$. A Clifford statistics is defined by a projective (double-valued) representation $R_C : S_N \to C_1 \subset C$ of the permutations by first-grade Clifford elements over the unit mode space $V_1$ (Finkelstein and Galiautdinov, 2001). To define $R_C$ we associate with the $n$th unit (for all $n = 1, \ldots, N$) a Clifford unit $\gamma_n$, and we represent every swap (transposition or 2-cycle) $(mn)$ of two distinct units by the difference $\pm(\gamma_n - \gamma_m) \in C_1$ of the associated Clifford units.

[11] In an earlier effort to dissect space–time, assuming multiple Fermi–Dirac statistics for the elements (Finkelstein, 1969, 1972, 1996). This false start led us eventually to the Clifford–Wilczek statistics (Finkelstein and Galiautdinov, 2001; Finkelstein and Rodriguez, 1984; Nayak and Wilczek, 1996; Wilczek, 1998b; Wilczek and Zee, 1982); an example of Clifford–Wilczek statistics is unwittingly developed in Finkelstein (1996, Chap. 16).

*Some useful terms*:

A *cliffordon* is a quantum with Clifford statistics.
A *squadron* is a quantum aggregate of cliffordons.
A *sib* is a quantum aggregate of bosons.
A *set* of quanta is an aggregate of fermions.
A *sequence* of quanta is a aggregate of Maxwell–Boltzmann quanta with a given sequential order (Finkelstein, 1969).

$R_C$ can be extended to a spinor representation of SO($N$) on a spinor space $\Sigma(N)$.

The symmetry group $G_U$ of the quantum kinematics for a universe $U$ of $N_U$ chronons is an orthogonal group

$$\begin{aligned} G_U &= \mathrm{SO}(N_{U+}, N_{U-}), \\ N_U &= N_{U+} + N_{U-}. \end{aligned} \tag{55}$$

The algebra of observables of $U$ is the simple finite-dimensional real Clifford algebra

$$C_U = \mathrm{Cliff}(V_1) = \mathrm{Cliff}[1, \gamma(1), \ldots, \gamma(N_U)] \tag{56}$$

generated by the $N_U$ Clifford units $\gamma(n)$, $n = 1, \ldots, N_U$, representing exchanges. The Clifford units $\gamma(n)$ span a vector space $V_1 \cong C_1$ of first-grade elements of $C_U$.

Within $C_U$ we shall construct a simplified Dirac–Heisenberg algebra

$$\breve{\mathcal{A}}_{\mathrm{DH}} = \mathcal{A}[\breve{i}, \breve{p}, \breve{x}, \tilde{\gamma}] \subset (V_1) \tag{57}$$

whose commutator Lie algebra is simple and which contracts to the usual Dirac–Heisenberg algebra $\mathcal{A}_{\mathrm{DH}}$ in the continuum limit. We factor $\breve{\mathcal{A}}_{\mathrm{DH}}$ into the Clifford product

$$\breve{\mathcal{A}}_{\mathrm{DH}} = \mathrm{Cliff}(N\mathbf{6_0}) = \mathrm{Cliff}((N-1)\mathbf{6_0}) \sqcup \mathrm{Cliff}(\mathbf{6_0}) \tag{58}$$

of two Clifford algebras, an “internal” algebra from the last hexad and an “external” algebra from all the others.

We designate our proposed simplifications of $\gamma$ and $i$, $\hat{p}$, $\hat{x}$, and $\hat{O}$ by $\tilde{\gamma}$ and $\breve{i}$, $\breve{p}$, $\breve{x}$ and $\breve{O}$. In the limit $\chi \to 0$ the tildes ˜ disappear and the breves ˘ become hats ˆ.

#### 3.1.3. *Relativistic Dirac–Heisenberg Algebra*

Each physical theory defines at least three algebras that should be simple:

1) the associative operator algebra of the system (Finkelstein, 1996, 1999),
2) the kinematical Lie algebra consisting of possible Hamiltonians, and
3) the symmetry Lie algebra of one preferred Hamiltonian.

There is no second quantization. But there is a second simplification, and a third, and so on, all of different kinds with different simplifiers. Each of the historic revolutions that guide us now introduced a simplifier, small on the scale of previous experience and therefore long overlooked, into the multiplication table and basis elements of one or more of these algebras, and so deformed a compound algebra into a simpler algebra that works better. Among these simplifiers are $c$, $G$, and $\hbar$.

Here we simplify the free Dirac equation and its underlying *Dirac–Heisenberg* (real unital associative) algebra

$$\mathcal{A}_{\mathrm{DH}} = \mathcal{A}_{\mathrm{D}} \otimes \mathcal{A}_{\mathrm{H}}. \tag{59}$$

Since we take the notion of dynamical process as primary in physics (Finkelstein, 1973), we express a measurement (observation, filtering, selection, yes–no experiment) as a special case of an interaction between observer and system. Therefore we first simplify the anti-Hermitian space–time and energy-momentum symmetry generators $\hat{p}^\mu$ and $\hat{x}_\nu$, not the associated Hermitian observables $p^\mu$, $x_\nu$. Then we simplify the Hermitian operators by multiplying the anti-Hermitian ones by a suitably simplified $i$.

The Dirac–Heisenberg algebra (59) is a tensor product of the Dirac and the relativistic Heisenberg algebras, in turn defined as follows:

*Relativistic Heisenberg algebra*. $\mathcal{A}_{\mathrm{H}} = \mathcal{A}[i, \hat{p}, \hat{x}]$ is generated by the imaginary unit $i$ and the space–time and energy-momentum translation generators $\hat{p}_\nu := ip_\nu \equiv -\hbar\partial/\partial x^\nu$ and $\hat{x}^\mu := ix^\mu$, subject to the relations

$$\begin{aligned}
[\hat{p}^\mu, \hat{x}^\nu] &= -i\hbar g^{\mu\nu},\\
[\hat{p}^\mu, \hat{p}^\nu] &= 0,\\
[\hat{x}^\mu, \hat{x}^\nu] &= 0,\\
[i, \hat{p}^\mu] &= 0,\\
[i, \hat{x}^\mu] &= 0,\\
i^2 &= -1.
\end{aligned} \tag{60}$$

Here $g^{\mu\nu}$ is the Minkowski metric, held fixed in this paper. The hats (on $\hat{p}$, for example) indicate that a factor $i$ has been absorbed to make the operator anti-Hermitian (Adler, 1995). The algebra $\mathcal{A}_{\mathrm{H}}$ has both the usual associative product and the Lie commutator product. As a Lie algebra $\mathcal{A}_{\mathrm{H}}$ is compound, Segal emphasized, containing the nontrivial ideal generated by the unit $i$.

The orbital Lorentz-group generators are

$$\hat{O}^{\mu\nu} := iO^{\mu\nu} = -i(\hat{x}^\mu\hat{p}^\nu - \hat{x}^\nu\hat{p}^\mu). \tag{61}$$

These automatically obey the usual relations

$$\begin{aligned}
[\hat{O}^{\mu\nu}, \hat{O}^{\lambda\kappa}] &= \hbar(g^{\mu\lambda}\hat{O}^{\nu\kappa} - g^{\nu\lambda}\hat{O}^{\mu\kappa} - g^{\mu\kappa}\hat{O}^{\nu\lambda} + g^{\nu\kappa}\hat{O}^{\mu\lambda}),\\
[\hat{x}^{\mu}, \hat{O}^{\nu\lambda}] &= \hbar(g^{\mu\nu}\hat{x}^{\lambda} - g^{\mu\lambda}\hat{x}^{\nu}),\\
[\hat{p}^{\mu}, \hat{O}^{\nu\lambda}] &= \hbar(g^{\mu\nu}\hat{p}^{\lambda} - g^{\mu\lambda}\hat{p}^{\nu}),\\
[i, \hat{O}^{\mu\nu}] &= 0.
\end{aligned} \tag{62}$$

*Dirac algebra*. $\mathcal{A}_{\mathrm{D}} = \mathcal{A}[\gamma_\mu]$ is generated by Dirac–Clifford units $\gamma_\mu$ subject to the familiar relations

$$\{\gamma_\nu, \gamma_\mu\} = 2g_{\nu\mu}. \tag{63}$$

As usual we write $\gamma_{\mu\nu\dots}$ for the antisymmetric part of the tensor $\gamma_\mu\gamma_\nu\dots$.

### 3.2. Simplification of the Relativistic Heisenberg Algebra

As already mentioned, field theory employs a compound field – space–time bundle with space–time for base and field–space for fiber, just as Galilean space–time is a four-dimensional bundle with $\mathbb{R}^3$ for base and $\mathbb{R}^1$ for fiber. The prototype is the covector field, where the fiber is the cotangent space to space–time, with coordinates that we designate by $p_\mu$.

*Main assumption. In experiments of sufficiently high resolution the space–time tangent bundle (or the Dirac–Heisenberg algebra) manifests itself as a simple quantum-field – space–time synthesis.*

The space–time variables $x^\mu$ and the tangent space variables $p_\mu$ unite into one simple construct, as space and time have already united. Now, however, the simplification requires an atomization, because the field variable actually derives from an atomic spin.

We first split the space–time tangent bundle into quantum cells. The minimum number of elements in a cell, for our simplification, is six: four for space–time and two for a complex or symplectic plane. We provisionally adopt the hexadic cell.[12]

$N$ hexads define a unit mode space $V_1 = N\mathbf{6_0}$. There are two possibilities here as to how to proceed. We can either use the Hexad Lemma described in Appendix E and work with hexadic cells whose variables commute with one another, or we can work directly with $6N$ anticommuting Clifford units of Cliff$(3N, 3N)$. In any case the external variables of the quantum probe related to different hexads as defined below will commute. The first possibility, however, might be the key to the derivation of the Maxwell–Boltzmann statistics for classical space–time points from a deeper quantum theory of elementary processes.

[12] Earlier work, done by Finkelstein and coworkers before our present stringent simplicity requirement, assumed a pentadic cell (Finkelstein and Rodriguez, 1984). This provided no natural correspondent for the energy–momentum operators (personal conversation with Finkelstein, 1999).

Operationally we do not deal with empty space–time. We explore space–time with one relativistic quantum spin-$\frac{1}{2}$ probe of rest mass $m \sim 1/\chi$. We express the usual spin operators $\gamma^{\mu}$, space–time position operators $x^{\mu}$, and energy–momentum operators $p_{\mu}$ of this probe as contractions of operators in the Clifford algebra Cliff$(3N, 3N)$.

We write the dynamics of the usual contracted, compound Dirac theory in manifestly covariant form, with a Poincaré-scalar Dirac operator

$$D = \gamma^{\mu} p_{\mu} - mc. \tag{64}$$

$D$ belongs to the algebra of operators on spinor-valued functions $\psi(x^{\mu})$ on space–time. Any physical spinor $\psi(x^{\mu})$ is to obey the dynamical equation

$$D\psi = 0. \tag{65}$$

We simplify the dynamical operator $D$, preserving the form of the dynamical equation (65).

### 3.2.1. *Simplifying External Variables*

The compound symmetry group for the Dirac equation is the covering group of the Poincaré group ISO ($\mathbb{M}$). We represent this as the contraction of a simple group SO(3, 3) acting on the spinor pseudo-Hilbert (ket) space of $6N$ Clifford generators $\gamma^{\omega}(n)$ ($\omega = 0, \ldots, 5; n = 1, \ldots, N$) of the orthogonal group SO$(3N, 3N)$. The size of the experiment fixes the parameter $N$.

As in Dirac one-electron theory (where the spin generators are represented by second-degree elements

$$\hat{S}^{\mu\nu} := \frac{\hbar}{4}[\gamma^{\mu}, \gamma^{\nu}] \equiv \frac{\hbar}{2}\gamma^{\mu\nu}, \quad \mu, \nu = 0, \ldots, 3 \tag{66}$$

of the Clifford algebra Cliff(1,3)), we use the second-degree elements of our Clifford algebra Cliff$V_1$ = Cliff$(N\mathbf{6_0})$ to represent anti-Hermitian generators of rotations, boosts, space–time, and energy–momentum shifts.

We associate the position and momentum axes with the $\gamma^4$ and $\gamma^5$ elements of the hexad respectively, so that an infinitesimal orthogonal transformation in the 45-plane couples momentum into position. This accounts for the symplectic symmetry of classical mechanics and the $i$ of quantum mechanics.

*Our choice of the simplified* $\breve{i}$, $\breve{x}^{\mu}$, *and* $\breve{p}^{\nu}$ *of the probe is*

$$\breve{i} \equiv \frac{1}{N-1}\sum_{n=1}^{N-1} \breve{i}(n) := \frac{1}{N-1}\sum_{n=1}^{N-1} \gamma^{45}(n),$$

$$\breve{x}^{\mu} \equiv \sum_{n=1}^{N-1} \breve{x}^{\mu}(n) := -\chi \sum_{n=1}^{N-1} \gamma^{\mu 4}(n),$$

$$\breve{p}^{\nu} \equiv \sum_{n=1}^{N-1} \breve{p}^{\nu}(n) := \phi \sum_{n=1}^{N-1} \gamma^{\nu 5}(n), \tag{67}$$

where $\chi$, $\phi$, and $N$ are simplifiers of our theory, and

$$\gamma^{\rho\sigma}(n) := \frac{1}{2}[\gamma^{\rho}(n), \gamma^{\sigma}(n)]. \tag{68}$$

To support this choice for the expanded generators we form the following commutation relations among them (*cf.* Doplicher, 2001; Snyder, 1947a,b):

$$\begin{aligned}
[\breve{p}^{\mu}, \breve{x}^{\nu}] &= -2\phi\chi(N-1)g^{\mu\nu}\breve{i}, \\
[\breve{p}^{\mu}, \breve{p}^{\nu}] &= -\frac{4\phi^2}{\hbar}\breve{L}^{\mu\nu}, \\
[\breve{x}^{\mu}, \breve{x}^{\nu}] &= -\frac{4\chi^2}{\hbar}\breve{L}^{\mu\nu}, \\
[\breve{i}, \breve{p}^{\mu}] &= -\frac{2\phi}{\phi(N-1)}\breve{x}^{\nu}, \\
[\breve{i}, \breve{x}^{\mu}] &= +\frac{2\chi}{\phi(N-1)}\breve{p}^{\mu}.
\end{aligned} \tag{69}$$

In (69),

$$\begin{aligned}
\breve{L}^{\mu\nu} &:= \frac{\hbar}{2}\sum_{n=1}^{N-1} \gamma^{\mu\nu}(n), \\
\breve{J}^{\mu\nu} &:= \frac{\hbar}{2}\sum_{n=1}^{N} \gamma^{\mu\nu}(n) \equiv L^{\mu\nu} + S^{\mu\nu}.
\end{aligned} \tag{70}$$

where $\breve{S}^{\mu\nu}$ is the Dirac spin operator (*cf.* (79)).

$J^{\mu\nu}$ obeys the Lorentz-group commutation relations

$$[\breve{J}^{\mu\nu}, \breve{J}^{\lambda\kappa}] = \hbar(g^{\mu\lambda}\breve{J}^{\nu\kappa} - g^{\nu\lambda}\breve{J}^{\mu\kappa} - g^{\mu\kappa}\breve{J}^{\nu\lambda} + g^{\nu\kappa}\breve{J}^{\mu\lambda}), \tag{71}$$

and generates a total Lorentz transformation of the variables $x^{\mu}$, $p_{\mu}$, $i$, and $S^{\mu\nu}$:

$$\begin{aligned}
[\breve{x}^{\mu}, \breve{J}^{\nu\lambda}] &= \hbar(g^{\mu\nu}\breve{x}^{\lambda} - g^{\mu\lambda}\breve{x}^{\nu}), \\
[\breve{p}^{\mu}, \breve{J}^{\nu\lambda}] &= \hbar(g^{\mu\nu}\breve{p}^{\lambda} - g^{\mu\lambda}\breve{p}^{\nu}), \\
[\breve{i}, \breve{J}^{\mu\nu}] &= 0, \\
[\breve{S}^{\mu\nu}, \breve{J}^{\lambda\kappa}] &= 0.
\end{aligned} \tag{72}$$

There is a mock orbital angular momentum generator of familiar appearance,

$$\breve{O}^{\mu\nu} := -\breve{i}(\breve{x}^{\mu}\breve{p}^{\nu} - \breve{x}^{\nu}\breve{p}^{\mu}). \tag{73}$$

$\breve{O}$ too obeys the Lorentz-group commutation relations. We relate $\breve{L}^{\mu\nu}$ and $\breve{O}^{\mu\nu}$ later.

To recover the canonical commutation relations for $\breve{x}^{\mu}$ and $\breve{p}_{\mu}$ we must impose

$$\chi\phi(N-1) \equiv \frac{\hbar}{2} \tag{74}$$

and assume that

$$\begin{aligned} \chi &\to 0, \\ \phi &\to 0, \\ N &\to \infty. \end{aligned} \tag{75}$$

Then the relations (69) reduce to the commutation relations (60) of the relativistic Heisenberg algebra $\mathcal{A}_H$ as required.

The three parameters $\chi$, $\phi$, $1/N$ are subject to one constraint $\chi\phi(N-1) = \hbar/2$ leaving two independent simplifiers. $N$ depends on the scope of the experiment, and is under the experimenter's control.

We can consider two contractions

1) either $\chi \to 0$ with $N$ constant,
2) or $N \to \infty$ with $\chi$ constant.

They combine into the continuum limit $\chi \to 0$, $N \to \infty$. We fix one simplifier $\chi$ by supposing that the mass of a probe approaches a finite limit as $N \to \infty$.

### *3.2.2. Condensation of i*

Since the usual complex unit $i$ is central and the simplified $\breve{i}$ is not, we suppose that the contraction process includes a projection that restricts the probe to one of the two-dimensional invariant subspaces of $\breve{i}$, associated with the maximum negative eigenvalue $-1$ of $\breve{i}^2$. This represents a condensation that aligns all the mutually commuting hexad spins $\gamma^{45}(n)$ with each other, so that

$$\gamma^{45}(n)\gamma^{45}(n') \to -1, \tag{76}$$

for any $n$ and $n'$. We call this the *condensation of i*.

### *3.2.3. Angular Momenta*

As was shown above, three sets of operators obeying Lorentz-group commutation relations appear in our theory. $\breve{L}^{\mu\nu}$ represents the simplified *orbital angular* *momentum* generators, $\breve{S}^{\mu\nu}$ represents the *spin* angular momentum, and $\breve{J}^{\mu\nu}$ represents the simplified *total angular momentum* generators. There is a mock orbital angular momentum $\breve{O}^{\mu\nu}$ (73).

In this section we show that $\hat{O} \to \hat{L}$ in the contraction limit.

Consider $\breve{O}^{\mu\nu}$. By definition,

$$
\begin{aligned}
\breve{O}^{\mu\nu} &= -(\breve{x}^{\mu}\breve{p}^{\nu} - \breve{x}^{\nu}\breve{p}^{\mu})\breve{i} \\
&= +\frac{\chi\phi}{N-1}\left(\sum_{n=1}^{N-1}\gamma^{\mu 4}(n)\sum_{n'=1}^{N-1}\gamma^{\nu 5}(n') - \sum_{n=1}^{N-1}\gamma^{\nu 4}(n)\sum_{n'=1}^{N-1}\gamma^{\mu 5}(n')\right)\sum_{m=1}^{N-1}\gamma^{45}(m) \\
&= +\frac{\chi\phi}{N-1}\sum_{n}\left(\gamma^{\mu 4}(n)\gamma^{\nu 5}(n) - \gamma^{\nu 4}(n)\gamma^{\mu 5}(n)\right)\sum_{m}\gamma^{45}(m) \\
&\quad + \frac{\chi\phi}{N-1}\sum_{n\neq n'}\left(\gamma^{\mu 4}(n)\gamma^{\nu 5}(n') - \gamma^{\nu 4}(n)\gamma^{\mu 5}(n') + \gamma^{\mu 4}(n')\gamma^{\nu 5}(n)\right. \\
&\quad \left. - \gamma^{\nu 4}(n')\gamma^{\mu 5}(n)\right)\left(\gamma^{45}(n) + \gamma^{45}(n')\right) + \frac{\chi\phi}{N-1}\sum_{n\neq n'}\left(\gamma^{\mu 4}(n)\gamma^{\nu 5}(n')\right. \\
&\quad \left. - \gamma^{\nu 4}(n)\gamma^{\mu 5}(n') + \gamma^{\mu 4}(n')\gamma^{\nu 5}(n) - \gamma^{\nu 4}(n')\gamma^{\mu 5}(n)\right)\sum_{m\neq n, m\neq n'}\gamma^{45}(m) \\
&= -\frac{2\chi\phi}{N-1}\sum_{n}\gamma^{\mu\nu}(n)\gamma^{45}(n)\sum_{m}\gamma^{45}(m) + \frac{\chi\phi}{N-1}\sum_{n\neq n'}\left(\gamma^{\mu 4}(n)\gamma^{\nu 5}(n')\right. \\
&\quad - \gamma^{\nu 4}(n)\gamma^{\mu 5}(n') + \gamma^{\mu 4}(n')\gamma^{\nu 5}(n) \\
&\quad \left. - \gamma^{\nu 4}(n')\gamma^{\mu 5}(n)\right)\sum_{m\neq n, m\neq n'}\gamma^{45}(m).
\end{aligned}
\tag{77}
$$

Thus, in the contraction limit (76)–(75) when condensation singles out the eigenspace of $\gamma^{45}(n)\gamma^{45}(n')$ with eigenvalue $-1$,

$$
\hat{O}^{\mu\nu} \longrightarrow \hat{J}^{\mu\nu} - \hat{S}^{\mu\nu} \equiv \hat{L}^{\mu\nu}, \tag{78}
$$

as asserted.

## 3.3. Simplification of the Dirac Equation

### *3.3.1. The Dynamics*

We simplify the Dirac–Heisenberg algebra $\mathcal{A}_{\mathrm{DH}}$ to $\breve{\mathcal{A}}_{\mathrm{DH}} := \mathrm{Cliff}(N\mathbf{6_0})$, the Clifford algebra of a large squadron of cliffordons.

To construct the contraction from $\breve{\mathcal{A}}_{\mathrm{DH}}$ to $\mathcal{A}_{\mathrm{DH}}$, we group the generators of Cliff$(3N, 3N)$ into $N$ hexads $\gamma^{\omega}(n)$ ($\omega = 0, \ldots, 5; n = 1, \ldots, N$). Each hexad algebra acts on eight-component real spinors in $\mathbf{8}_0$ (see Appendix E).

*Reminder*. Hexad $N$ is used for the spin of the quantum. The remaining $N - 1$ hexads provide the space–time variables.

Dirac's spin generators $\hat{S}^{\mu\nu}$ (66) simplify to the corresponding 16 components of the tensor

$$\breve{S}^{\omega\rho} := \frac{\hbar}{2}\gamma^{\omega\rho}(N), \tag{79}$$

where $\omega, \rho = 0, \ldots, 5$ and $\mu, \nu = 0, \ldots, 3$.

Then *the most natural choice for Dirac's dynamics* is

$$\tilde{D} := \frac{2\phi}{\hbar^2}\breve{S}^{\omega\rho}\breve{L}_{\omega\rho}, \tag{80}$$

where (*cf.* (70))

$$\breve{L}_{\omega\rho} := \frac{\hbar}{2}\sum_{n=1}^{N-1}\gamma_{\omega\rho}(n), \tag{81}$$

and

$$\frac{2\phi}{\hbar^2} = \frac{1}{\hbar\chi(N-1)}.$$

The proposed dynamical operator is invariant under SO(3, 3). This symmetry group incorporates and extends the SO(3, 2) symmetry possessed by Dirac's dynamics for an electron in de-Sitter space–time (Dirac, 1935).

$$D' = \frac{1}{\hbar R}\hat{S}^{\omega\rho}\hat{O}_{\omega\rho} - mc,$$

where $\hat{S}^{\omega\rho}$ and $\hat{O}_{\omega\rho}$ are the five-dimensional spinorial and orbital angular momentum generators and $R$ is the radius of the de-Sitter universe. That group unifies translations, rotations, and boosts, but not the $i$.

### *3.3.2. Reduction to the Poincaré Group*

We now assume a condensation that reduces SO(3, 3) into SO(1,3) × SO(2). Relative to this reduction, the $\tilde{D}$ of (80) breaks up into

$$\begin{aligned}\tilde{D} &= \frac{\phi}{2}\gamma^{\omega\rho}(N)\sum_{n}\gamma_{\omega\rho}(n) \quad \omega, \rho = 0, 1, \ldots, 5 \\ &= \phi\gamma^{\mu 5}(N)\sum_{n}\gamma_{\mu 5}(n) + \phi\gamma^{\mu 4}(N)\sum_{n}\gamma_{\mu 4}(n) + \phi\gamma^{\mu\nu}(N)\sum_{n}\gamma_{\mu\nu}(n)\end{aligned}$$

$$+\phi\gamma^{45}(N)\sum_n \gamma_{45}(n)$$
$$= \gamma^{\mu 5}\breve{p}_\mu - \frac{\phi}{\chi}\gamma^{\mu 4}\breve{x}_\mu + \frac{2\phi}{\hbar}\gamma^{\mu\nu}\breve{L}_{\mu\nu} + (N-1)\phi\gamma^{45}\breve{i}. \tag{82}$$

In the condensate all the operators $\gamma^{45}(n)\gamma_{45}(n')$ attain their minimum eigenvalue $-1$. Then

$$(N-1)\phi\gamma^{45}\breve{i} \longrightarrow -\frac{\hbar}{2\chi} \tag{83}$$

and the dynamics becomes

$$\tilde{D} = \gamma^{\mu 5}\breve{p}_\mu - \frac{\phi}{\chi}\gamma^{\mu 4}\breve{x}_\mu + \frac{2\phi}{\hbar}\gamma^{\mu\nu}\breve{L}_{\mu\nu} - m_\chi c, \tag{84}$$

with rest mass

$$m_\chi = \frac{\hbar}{2\chi c}. \tag{85}$$

Here we identify the usual Dirac gammas $\gamma^\mu$, for $\mu = 0, \ldots, 3$, of Cliff($\mathbb{M}$) with second-degree elements of the last hexad:

$$\gamma^\mu \cong \tilde{\gamma}^\mu := \gamma^{\mu 5}(N) \tag{86}$$

For sufficiently large $N$ this $\tilde{D}$ reduces to the usual Dirac dynamics.

We identify the mass $m_\chi$ with the $N$-independent mass $m$ of the Dirac equation for the most massive individual quanta that the condensate can propagate without meltdown, on the order of the top quark:

$$m_\chi \sim 10^2 \text{ GeV}, \quad \chi \sim 10^{-25} \text{ s}. \tag{87}$$

The universe is $\sim 10^{10}$ years old. This leads to an upper bound

$$N_{\text{Max}} \sim 10^{41}. \tag{88}$$

Here $\chi$ is independent of $N$ as $N \to \infty$ and that $\phi \sim 1/N \to 0$ as $N \to \infty$ even for finite $\chi$.

In experiments near the Higgs energy, $p \sim \hbar/\chi$. If we also determine $N$ by setting $x \sim N\chi$ then all four terms in (84) are of the same order of magnitude.

To estimate experimental effects, however, we must take gauge transformations into account. These transform the second term away in the continuum limit. This refinement of the theory is still in progress.

## 4. CONCLUSIONS

Like classical Newtonian mechanics, the Dirac equation has a compound (non-semisimple) invariance group. Its variables break up into three mutually commuting sets: the space–time – energy–momentum variables $(x^\mu,\ p_\mu)$, the spin variables $\gamma^\mu$, and the imaginary unit $i$.

To unify them we replace the space–time continuum by an aggregate of $M < \infty$ finite elements, chronons, described by spinors with $\sim 2^{M/2}$ components. Chronons have Clifford–Wilcezek statistics, whose simple operator algebra is generated by units $\gamma^m$, $m = 1, \ldots, M$. We express all the variables $x^\mu$, $p_\mu$, $\gamma^\mu$, and $i$ as polynomials in $\gamma^m$. We group the $M = 6N$ chronons into $N$ hexads for this purpose, corresponding to tangent spaces; the hexad is the least cell that suffices for this simplification. There are three simplifiers $\chi$, $\phi$, $1/N$, all approaching 0 in the continuum limit, subject to the constraint $\chi\phi(N-1) = \hbar/2$ for all $N$.

In the continuum limit the Dirac mass becomes infinite. In our theory, the finite Dirac masses in nature are consequences of a finite atomistic quantum space–time structure with $\chi > 0$.

The theory predicts a certain spin–orbit coupling $\gamma^{\mu\nu}L_{\mu\nu}$ not found in the Standard Model, and vanishing only in the continuum limit. The experimental observation of this spin–orbit coupling would further indicate the existence of a chronon.

In this theory, the spin we see in nature is a manifestation of the (Clifford) statistics of atomic elements of space–time, as Brownian motion is of the atomic elements of matter. As we improve our theory we will interpret better other indications of chronon structure that we already have, and as we improve our measuring techniques we shall meet more such signs.

## APPENDIX A: OPERATIONALITY AND THE GEOMETRICAL NATURE OF PHYSICS

A good physical theory should be based on the following.

*Operational Postulate*. If we do so-and-so, we will find such-and-such.

Generalizing these "doing and finding" actions (carried out by an experimenter) to the operations going on in Nature independent of any observer, and taking them as the basic units of our theory, we arrive at systems analogous to the ones frequently used in mathematics. By applying to these systems the rules of mathematical inference we derive new predictions that could be tested in experiment.

One of the most remarkable mathematical systems often used in physics is the system of geometry in which the basic structural element of paramount importance is an incidence basis (Mihalek, 1972). The incidence basis consists of two (incidence) classes, disjoint or not, often called the class of points $\mathcal{P}$ and the class of lines $\mathcal{L}$, and the incidence relation, usually denoted by $\circ$, between the elements of the two classes.

Traditionally the notion of incidence involves a definite rule of pairing of elements from $\mathcal{P}$ with elements from $\mathcal{L}$, or, more specifically, their ordered pairing.

To realize the ordered pairing, the notion of an ordered pair $(a, b)$ is introduced, where $a$ and $b$ are the elements of the pair with $a$ the first and $b$ the second. The set of all ordered pairs $(a, b)$, $a \in \mathcal{P}$, $b \in \mathcal{L}$ is called the Cartesian product of $\mathcal{P}$ and $\mathcal{L}$ and designated by $\mathcal{P} \times \mathcal{L}$.

The study of incidence, then, involves certain subset of $\mathcal{P} \times \mathcal{L}$ whose specification establishes an incidence relation between the elements of the two classes, in analogy to how it is done, for example, in elementary Euclidean geometry.

The properties of a particular incidence relation are called axioms of the corresponding incidence basis. Together they define a geometry.

It is a remarkable fact that both most important theories of contemporary physics, quantum theory and relativity, can be regarded as geometries with some definite forms of the incidence bases. Moreover, any physical theory obeying The Operational Postulate must have the form similar to some (generalized) geometry. This is because the postulate itself has the incidence structure: our "doings" and "findings" can be regarded as the elements of some incidence classes, and the correlation between the two may be viewed as an incidence relation.

Formulating all of physics in the form of an incidence basis is a tedious task. It involves considerable labor of stating and proving the theorems of the corresponding geometrical system and relating them back to the experience in the form of some definite operational procedures.

What is more important, however, is that trying to fit everything in the usual geometrical framework, would eventually lead to the same limitations that were encountered in the classical axiomatic method. This can be seen as follows:

As we know, one of the main problems of the classical axiomatic method is the "proof" of internal consistency of a given geometry, so that no mutually contradictory theorems can be deduced from the axioms of an incidence basis under consideration.

A general method of "solving" this problem is based on exhibiting a basis, called a consistency basis (a model or interpretation), so that every axiom of the original basis is converted into a "true" statement about the model (Mihalek, 1972; Nagel and Newman, 1960). For example, it is possible to show that the model for the Euclidean geometry can be grounded on the axiomatic system of elementary arithmetics, etc. Unfortunately in most cases, including this one, the method just shifts the problem of consistency of a geometry to the problem of consistency of the model itself. If the geometry is such that a finite model can be built whose incidence classes contain a finite number of elements, we may try to establish the consistency of the geometry by direct inspection of the model and determining whether its elements satisfy the original axioms. However for most of the axiomatic systems that are important in mathematics and used by physicists finite consistency bases cannot be constructed.

In this respect, we may regard a physical theory obeying The Operational Postulate as a consistency basis (a model, interpretation) of a geometry we choose to work with. Although from this perspective, of course, everything looks turned around—usually the theory, not the results of experiments, is regarded as a model—a closer look reveals that this is just an expression of our constant desire to use mathematics and its methods in discovering the workings of the world around us. Here the model is, in a sense, richer than the corresponding geometry, exactly how it is often in mathematics. While in mathematics the extra properties of the model might obcure the derivation of interesting theorems, in physics these are the assets, and their discovery constitutes the ultimate goal and purpose of science.

An alternative to the consistency basis method is the method of an absolute proof of consistency in which the consistency of the system is sought to be established without assuming the consistency of some other model system (Nagel and Newman, 1960). As far as physics is concerned this method is important in the mathematical part of the theory—deriving the predictions (see below)—as providing justification for the use of ordinary logic in manipulating the recorded results of experiments. However even this ambitious method has failed to solve the general problem of consistency. As Gödel showed, it is impossible in principle to establish the internal logical consistency of a large class of axiomatic systems—including elementary arithmetics—without adopting principles of reasoning so complicated that their own internal consistency is no less doubtful than that of the systems themselves.

Another important limitation of the classical axiomatic method discovered by Gödel is that any axiomatic system within which arithmetic can be developed is essentially incomplete, meaning that given any consistent set of axioms, there are true arithmetical statements that cannot be derived from that set.

It is of no surprise then that the chain

do experiment $\rightarrow$ generalize $\rightarrow$ idealize $\rightarrow$ axiomatize (and possibly modify by introducing additional elements to the incidence basis) $\rightarrow$ derive $\rightarrow$ translate into operational procedure $\rightarrow$ check by doing experiment,

would miss or distort some important physical content and introduce some inconsistencies of the geometry that will result in the impossibility of performing the suggested experiments, or simply lead to unverifiable predictions.

All theories of physics suffer in one way or another from the above-mentioned limitations of the very axiomatic method. New phenomena may always be discovered whose existence cannot be predicted (or disproved) within the axiomatic system, no matter how full the incidence relation of The Operational Postulate of Physics we choose. That is why the whole body of physics must be formulated in the operational terms. If then it turns out possible to construct an axiomatic system that helps us make new predictions, good—we accept it as a working theory. If not, we may be led to a new scheme (based on a method that is different from the axiomatic method) in terms of which physics would be formulated.

So far the usual mathematics with its axiomatic method has worked remarkably well. In particular, the system of geometry on which quantum mechanics is based has been very successful within its domain of applicability. Its basic "ingredients" have a well defined operational meaning. By itself quantum mechanics does not require any interpretation. It is its own interpretation.

Here, following Finkelstein (1996, 1999), we briefly summarize the quantum mechanical operational terminology which will be used throughout in this work.

We start with the kets, which represent sharp initial actions on the system under study. These initial actions are typically of the form "release from the source and then select with a filter."

Kets do not represent states of the system, contrary to popular belief. Kets represent what (kind of filtering) we do to the system in the beginning of each experiment. The notion of a state does not make much sense in quantum mechanics, especially if applied to one individual quantum in experiment.

Similarly, and dually, the bras represent final filtering actions followed by detection with an appropriate counter. Also, operators represent all possible operations on the system. The kets and bras are special kinds of operators. Mathematically they can be regarded as the elements of the minimal left and right ideals of system's operator algebra.

If we do a polarization experiment with a photon,

$$\text{(source} \to \text{initial polarizer)} \to \text{time evolution} \to \text{(final polarizer} \to \text{detector)},$$

no matter what we do to the photon after the initial action is completed, we will never be able to tell by what filter (vertical, horizontal, circular) it was selected initially. To find that out we would have to go back to the initial filter and look at it.

If the photon had a state, by "determining" it we would be able to tell unambiguously what initial filter had been used and what final filter will have to be used in order for the experiment to end up with a counter click. Such determination is possible in classical mechanics where the notion of the state is meaningful, but not in quantum mechanics.

It turns out that the superposition principle (which is a typical reason for retaining the nonoperational "state" terminology in quantum mechanics) can be naturally formulated for the initial actions. The initial actions can be viewed as the elements of a Hilbert space, and so all the usual mathematical formalism of quantum mechanics survives.

Thus, we again start with two spaces, the ket space $V$ of initial selective actions on the system and its dual space $V^\dagger$ of the final selective actions.

We have an adjoint that maps the two (see Appendix B). If the adjoint is positive-definite we get the usual quantum mechanics with the positive-definite metric.

We contract an initial ket with a final bra using that adjoint, to get the transition amplitude for the two-stage experiment of the form:

$$\text{(source} \rightarrow \text{initial selective act)} \rightarrow \text{(final selective act} \rightarrow \text{detector).}$$

*The operational meaning of the adjoint* is the following:

The final bra $\langle\psi| = |\psi\rangle^\dagger$ which is the adjoint of an initial ket $|\psi\rangle$, is such a final act that the transition $\langle\psi| \leftarrow |\psi\rangle$ is always allowed (every time we send a quantum it goes through both initial and final filters and the detector clicks). Problems might appear with an indefinite adjoint. In that case some transitions $\langle\psi| \leftarrow |\psi\rangle$ never happen because sometimes $\langle\psi \mid \psi\rangle = 0$.

Using the adjoint we could naturally introduce metric on both $V$ and $V^\dagger$, but its operational interpretation would be obscure. It is always better to keep the operational difference between initial (ket) and final (bra) spaces in mind and talk about the adjoint, not the metric

Classical mechanics can be easily cast into similar operational form by switching off superposition of actions.

And finally, *The superposition principle*:

An initial act $|\psi_3\rangle$ is a coherent superposition of initial acts $|\psi_1\rangle$ and $|\psi_2\rangle$,

$$|\psi_3\rangle = |\psi_1\rangle + |\psi_2\rangle,$$

if every final act $\langle\phi|$ that occludes $|\psi_1\rangle$ and $|\psi_2\rangle$ also occludes $|\psi_3\rangle$. In other words, if the transitions $\langle\phi| \leftarrow |\psi_1\rangle$ and $\langle\phi| \leftarrow |\psi_2\rangle$ never happen, then the transition $\langle\phi| \leftarrow |\psi_3\rangle$ (for the same actions $\langle\phi| \in V^\dagger$) never happens either. In Dirac's notation,

$$(\langle\phi|\psi_1\rangle = 0 \text{ AND } \langle\phi|\psi_2\rangle = 0) \Longrightarrow (\langle\phi|\psi_3\rangle = 0).$$

And dually for the superposition of final actions.

*Example*. Spin-1 particle in Stern–Gerlach experiment.

Here,

$$|\psi_1\rangle = |+1\rangle, \qquad |\psi_2\rangle = |-1\rangle, \qquad \langle\phi| = \langle 0|,$$

$$|\psi_3\rangle = |\psi_1\rangle + |\psi_2\rangle = |+1\rangle + |-1\rangle,$$

(both filters are opened), with

$$\langle 0 \mid +1\rangle = 0, \qquad \langle 0 \mid -1\rangle = 0 \quad \text{(transitions never happen).}$$

Then

$$\langle 0 \mid \psi_3\rangle = 0,$$

meaning that this transition never happens either. (Of course here all the vectors in question must be properly normalized.)

Thus in quantum mechanics we do not need states. We do not need nouns (what system *is*), as Finkelstein puts it, all we need are verbs (what we *do* to the system). In fact we *define* a particular system by what we can *do* to it.

By inventing nonoperational concepts like "state vector," "collapse of the wave function," etc., it is easy to drive ourselves into many contradictions and paradoxes. Keeping the operational meaning of quantum mechanics in mind, however, can help us avoid such pitfalls.

## APPENDIX B: ADJOINT OPERATOR

### B.1. Definition

Let $\mathbb{F}$ be a field of real ($\mathbb{R}$) or complex ($\mathbb{C}$) numbers and $V$ be a *right* $\mathbb{F}$-linear space with *some arbitrary* basis $\{e_a \mid a = 1, \ldots, \dim V\}$ in it.

An operator $\dagger$,

$$\dagger : V \to V^\dagger, \qquad \psi = e_a \psi^a \equiv (\psi^a) \mapsto \dagger\psi := \psi^\dagger = \psi_a^\dagger e^a \equiv (\psi^\dagger{}_a), \tag{B1}$$

from $V$ to its *left* $\mathbb{F}$-linear dual space $V^\dagger$ (with *some* basis $\{e_a \mid a = 1, \ldots, \dim V\}$) is said to be

- *singular* when
$$\exists \psi \neq 0 : \psi^\dagger = 0; \tag{B2}$$
- *symmetric* when
$$\phi^\dagger \cdot \psi = \psi^\dagger \cdot \phi; \tag{B3}$$
- *Hermitian-symmetric* when
$$\phi^\dagger \cdot \psi = (\psi^\dagger \cdot \phi)^C; \tag{B4}$$
- *definite* when
$$\forall \psi \neq 0,\ \psi^\dagger \cdot \psi \neq 0; \tag{B5}$$
- *positive (definite)* when
$$\forall \psi \neq 0,\ \psi^\dagger \cdot \psi > 0. \tag{B6}$$

Here

$$\phi^\dagger \cdot \psi := \phi^\dagger{}_a \psi^a \equiv \langle \phi \mid \psi \rangle \tag{B7}$$

means the *contraction* of $\phi^\dagger \in V^\dagger$ with $\psi \in V$ and $^C$ stands for the complex conjugation.

By definition, an *adjoint operator* is a mapping $\dagger : V \to V^\dagger$ that is antilinear, nonsingular, and Hermitian-symmetric. (When $\mathbb{F} = \mathbb{R}$, an adjoint operator is linear and symmetric.)

Defining

$$\dagger e_a = e_a{}^\dagger := M_{ba} e^b, \tag{B8}$$

we get

$$\psi^\dagger = \dagger(e_a \psi^a) := \psi^{aC}(\dagger e_a) = M_{ba}\psi^{aC} e^b, \tag{B9}$$

leading to

$$\psi^\dagger{}_b = M_{ba}\psi^{aC}. \tag{B10}$$

The contraction of $\psi^\dagger \in V^\dagger$ with $\phi \in V$ is then

$$\psi^\dagger \cdot \phi = M_{ba}\psi^{aC}\phi^b \equiv (M\psi^C)^{\mathrm{T}}\phi, \tag{B11}$$

$^{\mathrm{T}}$ meaning transposition.

The assumption of Hermitian symmetry of $\dagger$ implies

$$M_{ba} = M^C_{ab}, \tag{B12}$$

or, equivalently,

$$M = M^{CT} =: M^*. \tag{B13}$$

The matrix $M_{ab}$ of $\dagger$ (called the *transition metric*) is defined relative to some *arbitrary* bases of the corresponding spaces $V$ and $V^\dagger$.

*In general*, it is *not* true that $e_a{}^\dagger = e^a$. Rather, in physical applications the form of the transition metric is decided *operationally*, namely, relative to the properly defined actions belonging to some initial and final frames. For example, in the *standard nonrelativistic* quantum mechanics it is often possible to find the frames relative to which $M_{ab}$ is the identity matrix ($\dagger$ is positive-definite) and $e_a{}^\dagger = e^a$, so that $\psi^\dagger = (\psi^\dagger{}_a)$ is determined by the "usual" rule "transpose + complex-conjugate." However, in the more complicated situations $\dagger$ may be indefinite.

Because $\dagger$ is nonsingular, its (antilinear) inverse $\dagger^{-1} : V^\dagger \to V$ can be defined by the condition

$$\dagger^{-1} : \psi^\dagger \mapsto \dagger^{-1}\psi^\dagger := \psi. \tag{B14}$$

If we give the action of $\dagger^{-1}$ on the basis elements of $V^\dagger$,

$$\dagger^{-1} e_b := e_f M^{fb}, \tag{B15}$$

then

$$\begin{aligned} \dagger^{-1}\psi^\dagger &= \dagger^{-1}\left(\psi^\dagger_b e^b\right) := (\dagger^{-1} e^b)\psi^\dagger{}_b{}^C \\ &= (\dagger^{-1} e^b)\left(M_{ba}\psi^{aC}\right)^C \\ &= (\dagger^{-1} e^b) M_{ba}{}^C \psi^a \end{aligned}$$

$$= e_f M^{fb} M_{ba} \psi^a$$
$$:= e_a \psi^a, \tag{B16}$$

with

$$M^{fb} M_{ba} := \delta^f_a. \tag{B17}$$

Usually, $\dagger^{-1}$ is denoted by the same symbol $\dagger$, regarding an adjoint operation as involutory antiautomorphism of the action semigroup.

Now, any *linear* operator $A : V \overset{\text{linear}}{\longrightarrow} V$ acting on $V$ can be represented (relative to some basis $\{e_a\}$) by its matrix $A^b_a$:

$$A : e_a \mapsto Ae_a := e_b A^b_a. \tag{B18}$$

Then,

$$\psi \mapsto A\psi = A\left(e_a \psi^a\right) := e_b A^b_a \psi^a. \tag{B19}$$

$A$ can also be considered as acting (linearly) on the elements $\psi^\dagger$ of the dual space $V^\dagger$ by the rule

$$\psi^\dagger \mapsto \psi^\dagger A = \left(\psi^\dagger_a e^a\right) A$$
$$:= \psi^\dagger{}_a (e^a A)$$
$$:= \psi^\dagger{}_a A^a_b e^b. \tag{B20}$$

This allows us to write $A$ in the form

$$A = e_b \otimes A^b_a \, e^a, \tag{B21}$$

with $e_b$ and $e^a$ *now* being the elements of two *mutually dual* bases

$$e^a \cdot e_b := \delta^a_b. \tag{B22}$$

Acting with (B21) on the basis vector $e_a$ recovers (B18).

This form of writing allows us to consider $A$ as acting on *both* dual spaces, $V$ and $V^\dagger$, by the usual rule: vectors are being acted upon from the left and dual vectors from the right.

This also gives the *resolution of the identity* (in the *dual bases*),

$$\mathbf{1} = \sum_a e_a \otimes e^a. \tag{B23}$$

Note that when $e_a{}^\dagger = e^a$ we get the usual resolution of the identity of the standard quantum mechanics,

$$\mathbf{1} = \sum_a e_a \otimes e_a{}^\dagger. \tag{B24}$$

We may define the *adjoint of A*, denoted by $A^\dagger : V \to V$, as

$$\psi^\dagger \cdot A^\dagger \phi := (\phi^\dagger \cdot A\psi)^{\mathrm{C}} \quad (\forall \psi, \phi \in V). \tag{B25}$$

Direct calculation leads to

$$A^{\dagger}{}^a_b = M^{af} A^{\mathrm{CT}}{}^f_g M_{gb}, \tag{B26}$$

or

$$A^\dagger = M^{-1} A^* M. \tag{B27}$$

This formula is valid with respect to arbitrary bases of $V$ and $V^\dagger$.

We also define the *unitary* operator $U$ acting on the vector space $V$ (or $V^\dagger$) as an operator obeying

$$U^\dagger U = U U^\dagger = \mathbf{1}, \tag{B28}$$

leading to the condition

$$(U\psi)^\dagger \cdot U\phi = \psi^\dagger \cdot \phi. \tag{B29}$$

The preceding consideration can be generalized from vector spaces to *modules*. Instead of the operator C acting on the complex field $\mathbb{C}$ we must now consider an appropriate *anti-automorphism* of the corresponding *ring* over which the module is built. As an example of such generalization we can mention a module over the division ring of quaternions with the usual quaternion conjugation.

## B.2. The Van der Waerden Notation

In spinorial general relativity the notion of adjoint is generalized to include the antilinear mappings. For completeness we give a brief summary of the special *dot notation* developed for such cases by Van der Waerden.

Generalizing from the symmetric case, an *adjoint operator* (symmetric or antisymmetric) is a mapping

$$\mathbf{C} : V \to \dot{V}, \qquad \psi = e_a \psi^a \equiv (\psi^a) \mapsto \mathbf{C}\psi := \dot{\psi} = \psi_{\dot{a}} e^{\dot{a}} \equiv (\psi_{\dot{a}}), \tag{B30}$$

which is *anti*linear, nonsingular, and Hermitian-(anti)symmetric. (When $\mathbb{F} = \mathbb{R}$, an adjoint operator is linear and (anti)symmetric.)

Defining

$$\mathbf{C} e_a \equiv \dot{e}_a := C_{\dot{b}a} e^{\dot{b}}, \tag{B31}$$

we get

$$\dot{\psi} = \mathbf{C}(e_a \psi^a) = \bar{\psi}^a (\mathbf{C} e_a) := C_{\dot{b}a} \bar{\psi}^a e^{\dot{b}}, \tag{B32}$$

leading to

$$\psi_{\dot{b}} := C_{\dot{b}a}\overline{\psi}^a. \tag{B33}$$

The contraction of $\dot{\psi} \in \dot{V}$ with $\phi \in V$ is then

$$\begin{aligned} \dot{\psi} \cdot \phi &= C_{\dot{b}a}\overline{\psi}^a(e^{\dot{b}} \cdot e_f)\phi^f = C_{\dot{b}a}\overline{\psi}^a\delta^{\dot{b}}_f\phi^f \\ &\equiv C_{fa}\overline{\psi}^a\phi^f. \end{aligned} \tag{B34}$$

Here,

$$(e^{\dot{b}} \cdot e_f) := \delta^{\dot{b}}_f, \tag{B35}$$

and

$$C_{fa} := C_{\dot{b}a}\delta^{\dot{b}}_f. \tag{B36}$$

The assumption of Hermitian (anti)symmetry of $\mathbf{C}$ implies

$$C_{fa} := (-)\overline{C}_{af}. \tag{B37}$$

Because $\mathbf{C}$ is nonsingular, its (antilinear) inverse $\mathbf{C}^{-1} : \dot{V} \to V$ can be defined by the condition

$$\mathbf{C}^{-1} : \dot{\psi} \mapsto \mathbf{C}^{-1}\dot{\psi} := \psi. \tag{B38}$$

We have

$$\begin{aligned} \psi &= \mathbf{C}^{-1}\dot{\psi} \\ &= \mathbf{C}^{-1}\big(\psi_{\dot{a}}e^{\dot{a}}\big) \\ &= (\mathbf{C}^{-1}e^{\dot{a}})\overline{\psi}_{\dot{a}} \\ &= e_b C^{b\dot{a}}\overline{\psi}_{\dot{a}}, \end{aligned} \tag{B39}$$

where we have defined

$$\mathbf{C}^{-1}e^{\dot{a}} := e_b C^{b\dot{a}}. \tag{B40}$$

Correspondingly,

$$\psi^b = C^{b\dot{a}}\overline{\psi}_{\dot{a}}. \tag{B41}$$

We also have

$$C^{b\dot{a}}C_{\dot{a}f} = \delta^b_f, \qquad C_{\dot{a}f}C^{f\dot{b}} = \delta^{\dot{a}}_{\dot{b}}. \tag{B42}$$

In the spinor algebra of special relativity developed in Rumer and Fet (1977), $V = \mathbb{C}^2$, and there is defined the *main antisymmetric bilinear form* on $V = \mathbb{C}^2$ by

$$G(\psi, \phi) := (\mathbf{C}\psi \mid \phi) \equiv (\dot{\psi} \mid \phi) = \overline{C}_{fa}\psi^a\phi^f. \tag{B43}$$

Similarly, on $\dot{V} = \dot{\mathbb{C}}^2$,

$$\dot{G}(\dot{\psi}, \dot{\phi}) := (\dot{\psi} \mid \mathbf{C}^{-1}\dot{\phi})^* \equiv (\dot{\psi} \mid \phi)^* = \bar{C}^{\dot{a}\dot{f}} \psi_{\dot{a}} \phi_{\dot{f}}. \tag{B44}$$

Thus, $C_{ab}$ is antisymmetric and can be written as

$$(C_{fa}) := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{B45}$$

Also,

$$(C^{\dot{f}\dot{a}}) := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{B46}$$

## APPENDIX C: IDEALS

Very often an algebra can be constructed from smaller algebras by some rules of assembling them (see, e.g., Benn and Tucker, 1987, p. 317). One particularly relevant to our theory example of such construction is the *direct sum* of two algebras, when an algebra $\mathcal{A} = \mathcal{B} \oplus \mathcal{C}$, as a vector space, is a direct sum of the vector spaces $\mathcal{B}$ and $\mathcal{C}$, and $\mathcal{B} \bullet \mathcal{C} = \mathcal{C} \bullet \mathcal{B} = 0$, where $\bullet$ indicates the product on $\mathcal{A}$, be it associative, commutator (Lie), or else.[13] An algebra that can be written as a direct sum of several algebras is called *reducible*.

Reducible algebras contain invariant subalgebras, also knows as ideals. There are different kinds of ideals: left, right, or two-sided.

A *left ideal* is a *subspace* $\mathcal{I} \subset \mathcal{A}$ such that $\mathcal{A} \bullet \mathcal{I} \subset \mathcal{I}$.
A *right ideal* is a *subspace* $\mathcal{I} \subset \mathcal{A}$ such that $\mathcal{I} \bullet \mathcal{A} \subset \mathcal{I}$.
A *two-sided ideal* is a *subspace* $\mathcal{I} \subset \mathcal{A}$ such that $\mathcal{A} \bullet \mathcal{I} \bullet \mathcal{A} \subset \mathcal{I}$.

It is clear that all these ideals are also *subalgebras* of $\mathcal{A}$. Moreover, if $\mathcal{A} = \mathcal{B} \oplus \mathcal{C}$ then *both* $\mathcal{B}$ and $\mathcal{C}$ are its *two-sided* ideals.[14] As to the one-sided ideals, it is possible that, for example, $\mathcal{B}$ is such an ideal, but $\mathcal{C}$ is not.[15]

[13] For associative algebras we write $\mathcal{BC} = \mathcal{CB} = 0$; for Lie algebras we write $[\mathcal{B}, \mathcal{C}] = 0$.

[14] **Proof**

$$\begin{aligned}(\mathcal{A} \bullet \mathcal{B}) \bullet \mathcal{A} &= ((\mathcal{B} \oplus \mathcal{C}) \bullet \mathcal{B}) \bullet (\mathcal{B} \oplus \mathcal{C}) \\ &= ((\mathcal{B} \bullet \mathcal{B}) \oplus (\mathcal{C} \bullet \mathcal{B})) \bullet (\mathcal{B} \oplus \mathcal{C}) \\ &= (\mathcal{B} \bullet \mathcal{B}) \bullet (\mathcal{B} \oplus \mathcal{C}) \\ &= ((\mathcal{B} \bullet \mathcal{B}) \bullet \mathcal{B}) \oplus ((\mathcal{B} \bullet \mathcal{B}) \bullet \mathcal{C})) \subset \mathcal{B}. \end{aligned} \tag{C1}$$

[15] The relativistic Heisenberg algebra (60) is an example, where $i$ generates a one-sided ideal, but $\hat{p}$ and $\hat{x}$ do not.

Let us now suppose $\mathcal{A} = \mathcal{B} + \mathcal{C}$ as a vector space, and let us define an equivalence relation in $\mathcal{A}$ by $a \sim b$ if $a = b + c$ where $c \in \mathcal{C}$. Denote the equivalence class of $a$ in the usual way by $[a]$.

The equivalence classes so defined can be made into a vector space with the rules of addition and scalar multiplication as follows:

$$[a] + [b] = [a + b], \tag{C2}$$

and

$$\lambda[a] = [\lambda a]. \tag{C3}$$

The question arises whether these equivalence classes can also be made into an algebra. An obvious choice for multiplication is

$$[a][b] = [a \bullet b]. \tag{C4}$$

However, if $c_1, c_2 \in \mathcal{C}$ then

$$\begin{aligned} [a][b] &= [a + c_1][b + c_2] \\ &= [(a + c_1) \bullet (b + c_2)] \\ &= [a \bullet b + a \bullet c_2 + c_1 \bullet b + c_1 \bullet c_2] \\ &= [a \bullet b + (a \bullet c_2 + c_1 \bullet b + c_1 \bullet c_2)], \end{aligned} \tag{C5}$$

which means that $(a \bullet c_2 + c_1 \bullet b + c_1 \bullet c_2) \in \mathcal{C}$. This is true if, and only if, $\mathcal{C}$ is a *two-sided ideal* of $\mathcal{A}$. The algebra of equivalent classes so constructed is called *quotient algebra* of $\mathcal{A}$ modulo $\mathcal{C}$, which is denoted by $\mathcal{A}/\mathcal{C}$.

This is precisely the way in which many important algebras (including the tensor, the Clifford, etc., algebras) are defined in modern mathematics. We also use this method to define various forms of quantification.

## APPENDIX D: LOCALIZATION PROBLEM

### D.1. The Planck Limit: Measuring a Field at a Point

Gravity and quantum theory provide a well-known qualitative lower bound to the size ($\sim\tau_\mathrm{p}$) of a space–time cell over which an average field of any kind can be measured with arbitrary precision.

*Argument*

*Step 1*: Localization of the field meter (say, a test particle) within a space–time cell $\sim (c\tau_\mathrm{p})^4$. By uncertainty relation,

$$\Delta\varepsilon\tau_\mathrm{p} = \hbar.$$

*Step 2*: Schwarzschild radius associated with $\Delta\varepsilon$ is

$$R_{\rm Sch} = \frac{2G\Delta M}{c^2} = \frac{2G\Delta\varepsilon}{c^4} = \frac{2G\hbar}{c^4\tau_{\rm p}}.$$

*Step 3*: To avoid formation of the horizon, set

$$R_{\rm Sch} < c\tau_{\rm p}.$$

*Step 4*: This leads to the Planck limit,

$$\tau_{\rm p} > \sqrt{\frac{2G\hbar}{c^5}} = 7.6 \times 10^{-44}\ {\rm s}, \tag{D1}$$

which corresponds to the Planck energy

$$\varepsilon_{\rm p} = \hbar/\tau_{\rm p} = 1.4 \times 10^9\ {\rm J} = 8.7 \times 10^{18}\ {\rm GeV},$$

Here,
$c = 2.998 \times 10^8$ m/s
$G = 6.673 \times 10^{-11}$ m$^3$/kg s$^2$
$\hbar = 1.054 \times 10^{-34}$ J s
1 GeV $= 1.602 \times 10^{-10}$ J $= 1.783 \times 10^{-27}$ kg
1 GeV$^{-1}$ $= 1.973 \times 10^{-14}$ cm $= 0.1973$ fm

## D.2. The Many-Cell Horizon Limit: Measuring a Field Over a Region of Space–Time

The validity of QFT not only requires that the field at one space–time point can be measured, but also that the field at *every* point of an experimental region at one time-instant be measured!

*Argument*

*Step 1*: Localization of the field meter (say, a test particle) within a space–time cell $\sim (c\tau)^4$. By uncertainty relation,

$$\Delta\varepsilon\tau = \hbar.$$

*Step 2*: If the measurement is performed over a cube of scale $\sim T$, then there will be

$$N \sim (T/\tau)^3$$

cells.

*Step 3*: The total uncertainty in energy is

$$\Delta E = N\Delta\varepsilon = \hbar\frac{T^3}{\tau^4}.$$

*Step 4*: Schwarzschild radius associated with $\Delta E$ is

$$R_{\mathrm{Sch}} = \frac{2G\Delta M}{c^2} = \frac{2G\Delta E}{c^4} = c\tau_{\mathrm{p}}^2 \frac{T^3}{\tau^4}.$$

*Step 5*: To avoid formation of the horizon, set

$$R_{\mathrm{Sch}} < cT.$$

*Step 6*: This leads to the many-cell horizon limit,

$$\tau > \sqrt{\tau_{\mathrm{p}} T} \gg \tau_{\mathrm{p}}. \tag{D2}$$

We estimate this for one of the most precise tests of QED, the Lamb shift, which is on the order of

$$\Delta E_{\mathrm{Lamb}} \sim 10^{-5}\ \mathrm{eV} \sim 10^{-24}\ \mathrm{J}.$$

This corresponds to

$$T_{\mathrm{Lamb}} \sim \hbar/\Delta E_{\mathrm{Lamb}} \sim 10^{-10}\ \mathrm{s}.$$

The localization limit is then

$$\tau_{\mathrm{Lamb}} > \sqrt{\tau_{\mathrm{p}} T_{\mathrm{Lamb}}} \sim 3 \times 10^{-27}\ \mathrm{s}, \tag{D3}$$

which is 17 orders of magnitude greater than Planck's limit, and about 100 times smaller than the chronon size associated with the top quark. This corresponds to energies on the order of 1000 GeV, which is within the reach of the next generation accelerators.

*Conclusion: In the new theory the localization limit must be present from the outset. This leads to the idea of a chronon.*

## APPENDIX E: THE SQUAD LEMMA

Here we reduce a squadron to a sequence of smaller squadrons. This reduces a single Clifford–Wilczek quantification to two quantifications in succession, one Clifford–Wilczek and one Maxwell–Boltzmann. This means that one quantification can replace the two needed for standard physics.

By a *squad* we mean a squadron whose top element $\gamma^{\uparrow}$ has positive signature ($\gamma^{\uparrow 2} = +1$) and is noncentral. A squad must have an even number of units. The possible signatures depend on the number of units. Any six independent units of neutral total signature comprise a squad. Any eight independent units of any signature $\sigma$ comprise a squad $\mathbf{8}_{\sigma}$. Eight is the least nontrivial number with this property. $N\mathbf{8}_{\sigma}$ is also a squad for any positive integer $N$.

If $\mathcal{B}, \mathcal{C} \subset \mathcal{A}$ are two subalgebras of the (real associative unital) algebra $\mathcal{A}$, we define the *product* subalgebras $\mathcal{BC}$ as the span of the set of algebraic products $\{bc \mid b \in \mathcal{B}, c \in \mathcal{C}\}$.

If $\mathcal{B} \cap \mathcal{C} = \mathbf{1}$ and $\forall b \in \mathcal{B}\ \forall c \in \mathcal{C} \mid bc = cb$, then $\mathcal{B}C = \mathcal{B} \otimes \mathcal{C}$, the *tensor product* of the two algebras.

**Squad Lemma** (Budinich and Trautman, 1988; Lounesto, 1997; Porteous, 1995; Snygg, 1997)**.** *If $\mathcal{P}$ is the mode space of a squad then*

$$\mathrm{Cliff}(N\mathcal{P}) \cong_{\dagger} \bigotimes_{1}^{N} \mathrm{Cliff}(\mathcal{P}). \tag{E1}$$

In other words, a Clifford product of $N$ squads is algebraically †-isomorphic to a Maxwell–Boltzmann sequence of those squads. In this way Clifford statistics naturally generates Maxwell–Boltzmann statistics for its squads.

Maxwell–Boltzmann statistics then reduces to Fermi, Bose, and all the parastatistics. This seems adequate for much of physics.

The construction of this isomorphism resembles the well-known Jordan–Wigner construction of higher-dimensional spin representations. We repeat it in the present context for convenient reference.

For definiteness we exhibit the construction for a neutral hexad $\mathbf{6_0}$. We label the $6N$ generators $i_\omega(n)$ of Cliff($N\mathbf{6_0}$),

$$\mathrm{Cliff}(N\mathbf{6_0}) = \mathrm{Cliff}[i_5(N), \ldots, i_0(N), \ldots, i_5(1), \ldots, i_0(1)], \tag{E2}$$

by two indices, an *internal* hexad index $\omega = 0, 1, 2, \ldots, 5$ and an *external* hexad index $n = 1, 2, \ldots, N$.

The generators $i_\omega(n)$ obey the usual Clifford algebraic relations

$$\{i_\omega(n), i_\rho(n')\} = +2\delta(n, n')G_{\omega\rho}(n), \tag{E3}$$

with

$$(G_{\omega\rho}) = \begin{bmatrix} (g_{\mu\nu}) & 0 \\ 0 & (\delta_{\alpha\beta}) \end{bmatrix}, \qquad (g_{\dot{\mu}\nu}) = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$(\delta_{\alpha\beta}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{E4}$$

and are symmetric with respect to the metric $G_{\omega\rho}$:

$$i_\omega(n)^\dagger = +i_\omega(n). \tag{E5}$$

Define the top element of each hexad,

$$i^\uparrow(n) := i_5(n) \ldots i_1(n) i_0(n). \tag{E6}$$

As we have already indicated,

$$
\begin{aligned}
(i^{\uparrow}(n))^2 &= +1,\\
(i^{\uparrow}(n))^{\dagger} &= -i^{\uparrow}(n),\\
[i^{\uparrow}(n), i^{\uparrow}(n')] &= 0,\\
[i^{\uparrow}(n), i_{\omega}(n')] &= 0 \quad \text{for } n \neq n',\\
\{i^{\uparrow}(n), i_{\omega}(n)\} &= 0.
\end{aligned}
\tag{E7}
$$

We now define *local units* $\Gamma_{\omega}(n)$ as the Clifford products

$$
\Gamma_{\omega}(n) := (-1)^{n+1} i_{\omega}(n) \overleftarrow{\prod_{m<n}} i^{\uparrow}(m), \tag{E8}
$$

ordered with $m$ increasing from left to right. Then

$$
\begin{aligned}
[\Gamma_{\omega}(n), \Gamma_{\rho}(n')] &= 0, \quad \text{for } n \neq n';\\
\{\Gamma_{\omega}(n), \Gamma_{\rho}(n)\} &= +2G_{\omega\rho}(n),
\end{aligned}
\tag{E9}
$$

and

$$
\Gamma_{\omega}(n)^{\dagger} = +\Gamma_{\omega}(n) \tag{E10}
$$

is Hermitian with respect to $G_{\omega\rho}(n)$.

The local units generate the same Clifford algebra Cliff($N\mathbf{6_0}$) as the original pre-local units $i_{\omega}(n)$. It then follows that

$$
\text{Cliff}(N\mathbf{6_0}) \cong_{\dagger} \underbrace{\text{Cliff}(\mathbf{6_0}) \otimes \ldots \otimes \text{Cliff}(\mathbf{6_0})}_{N \text{ times}} \equiv \bigotimes_{n=1}^{N} \text{Cliff}(\mathbf{6_0}(n)) \tag{E11}
$$

as †-algebras. □

This case of the squad lemma is the *hexad lemma*. It shows how a huge squadron of prelocal anticommuting elementary processes can break up into a Maxwell–Boltzmann sequence of commuting hexads of local operations—the seed of classical space–time. Similar results obtain for any squad.

As a result, each term $\mathbf{6_0}$ of $V = N\mathbf{6_0}$ has a Clifford algebra Cliff(3, 3) associated with it, whose spinors have eight real components, forming an $\mathbf{8_0}$.[16] The spinors of $V$ form the spinor space $\Sigma(N\mathbf{6_0}) = \otimes_1^N \mathbf{8}_0 = \mathbf{8}_0^N$.

[16] Eight-component spinors have also been used in physics by Penrose (1971), Robson and Staudte (1996a), Staudte (1996b), and Lunsford (private communication, ****), though not to unify spin with space–time.

## APPENDIX F: PROJECTIVE REPRESENTATIONS OF THE PERMUTATION GROUP

Let us now turn to projective representations of the symmetric (permutation) groups that have long been known to mathematicians, but received little attention from physicists. Such representations were overlooked in physics much like projective representations of the rotation groups were overlooked in the early days of quantum mechanics.

For convenience, following Schur (1911), Karpilovsky (1985), and Hoffman and Humphreys (1992) (*cf.* also Wilczek, 1998a,b), we briefly recapitulate the main results of Schur's theory.

One especially useful presentation of the symmetric group $S_N$ on $N$ elements is given by

$$S_N = \langle t_1, \ldots, t_{N-1} : t_i^2 = 1, (t_j t_{j+1})^3 = 1, t_k t_l = t_l t_k \rangle,$$
$$\mathbf{1} \le i \le N-1, \quad 1 \le j \le N-2, \quad k \le l-2. \tag{F1}$$

Here $t_i$ are transpositions,

$$t_1 = (12), \quad t_2 = (23), \ldots, \quad t_{N-1} = (N - \mathbf{1}N). \tag{F2}$$

Closely related to $S_N$ is the group $\tilde{S}_N$,

$$\tilde{S}_N = \langle z, t_1', \ldots, t_{N-1}' : z^2 = 1, zt_i' = t_i' z, t_i'^2 = z, (t_j' t_{j+1}')^3 = z, t_k' t_l' = z t_l' t_k' \rangle,$$
$$1 \le i \le N-1, \quad \mathbf{1} \le j \le N-2, \quad k \le l-2. \tag{F3}$$

A celebrated theorem of Schur (1911) states the following:

(i) The group $\tilde{S}_N$ has order $2(n!)$.
(ii) The subgroup $\{1, z\}$ is central, and is contained in the commutator subgroup of $\tilde{S}_N$, provided $n \ge 4$.
(iii) $\tilde{S}_N/\{\mathbf{1}, z\} \simeq \tilde{S}_N$.
(iv) If $N < 4$, then every projective representation of $S_N$ is projectively equivalent to a linear representation.
(v) If $N \ge 4$, then every projective representation of $S_N$ is projectively equivalent to a representation $\rho$,

$$\rho(S_N) = \langle \rho(t_1), \ldots, \rho(t_{N-1}) : \rho(t_i)^2 = z, (\rho(t_j)\rho(t_{j+1}))^3 = z,$$
$$\rho(t_k)\rho(t_l) = z\rho(t_l)\rho(t_k)\rangle,$$
$$1 \le i \le N-1, \quad 1 \le j \le N-2, \quad k \le l-2, \tag{F4}$$

where $z = \pm 1$. In the case $z = +1$, $\rho$ is a linear representation of $S_N$.

The group $\tilde{S}_N$ (F3) is called the *representation group* for $S_N$.

The most elegant way to construct a *projective* representation $\rho(S_N)$ of $S_N$ is by using the complex Clifford algebra $\mathrm{Cliff}_{\mathbb{C}}(V, g) \equiv \mathcal{C}_N$ associated with the real vector space $V = N\mathbb{R}$,

$$\{\gamma_i, \gamma_j\} = -2g(\gamma_i, \gamma_j). \tag{F5}$$

Here $\{\gamma_i\}_{i=1}^N$ is an orthonormal basis of $V$ with respect to the symmetric bilinear form

$$g(\gamma_i, \gamma_j) = +\delta_{ij}. \tag{F6}$$

Clearly, any subspace $\bar{V}$ of $V = N\mathbb{R}$ generates a subalgebra $\mathrm{Cliff}_{\mathbb{C}}(\bar{V}, \bar{g})$, where $\bar{g}$ is the restriction of $g$ to $\bar{V} \times \bar{V}$. A particularly interesting case is realized when $\bar{V}$ is

$$\bar{V} := \left\{ \sum_{k=1}^N \alpha^k \gamma_k : \sum_{k=1}^N \alpha_k = 0 \right\} \tag{F7}$$

of codimension one, with the corresponding subalgebra denoted by $\bar{\mathcal{C}}_{N-1}$ (Hoffman and Humphreys, 1992).

If we consider a special basis $\{t'_k\}_{k=1}^{N-1} \subset \bar{V}$ (which is *not* orthonormal) defined by

$$t'_k := \frac{1}{\sqrt{2}}(\gamma_k - \gamma_{k+1}), \quad k = 1, \ldots, N-1, \tag{F8}$$

then the group generated by this basis is isomorphic to $\tilde{S}_N$. This can be seen by mapping $t_i$ to $t'_i$ and $z$ to $-1$, and by noticing that

1) For $k = 1, \ldots, N-1$:

$$\begin{aligned} t_k'^{\,2} &= \frac{1}{2}(\gamma_k - \gamma_{k+1})(\gamma_k - \gamma_{k+1}) \\ &= \frac{1}{2}\left(\gamma_k^2 + \gamma_{k+1}^2\right) \\ &= -1; \end{aligned} \tag{F9}$$

2) For $N - 2 \geq j$:

$$\begin{aligned} t'_j t'_{j+1} t'_j &= \frac{1}{2\sqrt{2}}(\gamma_j - \gamma_{j+1})(\gamma_{j+1} - \gamma_{j+2})(\gamma_j - \gamma_{j+1}) \\ &= \frac{1}{\sqrt{2}}(\gamma_j - \gamma_{j+2}), \end{aligned} \tag{F10}$$

and

$$t'_{j+1} t'_j t'_{j+1} = \frac{1}{\sqrt{2}}(\gamma_{j+2} - \gamma_j), \tag{F11}$$

so

$$\left(t'_j t'_{j+1}\right)^3 = \frac{1}{2}(\gamma_j - \gamma_{j+2})^2$$
$$= -1; \tag{F12}$$

3) For $N - 1 \geq m > k + 1$:

$$t'_k t'_m = +\frac{1}{2}(\gamma_k - \gamma_{k+1})(\gamma_m - \gamma_{m+1})$$
$$= -\frac{1}{2}(\gamma_m - \gamma_{m+1})(\gamma_k - \gamma_{k+1})$$
$$= -t'_m t'_k. \tag{F13}$$

One choice for the matrices is provided by the following construction (Brauer and Weyl, 1935):

$$\gamma_{2k-1} = \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes (i\sigma_1) \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1},$$
$$\gamma_{2k} = \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes (i\sigma_2) \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \quad k = 1, 2, 3, \ldots, M, \tag{F14}$$

for $N = 2M$. Here $\sigma_1$, $\sigma_2$ occur in the $k$th position, the product involves $M$ factors, and $\sigma_1$, $\sigma_2$, $\sigma_3$ are the Pauli matrices.

If $N = 2M + 1$, we first add one more matrix,

$$\gamma_{2M+1} = i\sigma_3 \otimes \cdots \otimes \sigma_3 \quad (M \text{ factors}), \tag{F15}$$

and then define

$$\Gamma_{2k-1} := \gamma_{2k-1} \oplus \gamma_{2k-1},$$
$$\Gamma_{2k} := \gamma_{2k} \oplus \gamma_{2k},$$
$$\Gamma_{2M+1} := \gamma_{2M+1} \oplus (-\gamma_{2M+1}). \tag{F16}$$

The representation $\rho(S_N)$ so constructed is reducible. An irreducible module of $\bar{\mathcal{C}}_{N-1}$ restricts that representation to the irreducible representation of $\tilde{S}_N$, since $\{t'_k\}_{k=1}^{N-1}$ generates $\bar{\mathcal{C}}_{N-1}$ as an algebra (Hoffman and Humphreys, 1992).

## APPENDIX G: APPLICATIONS TO THE THEORY OF THE FQHE

In this appendix I will describe my very first attempt at understanding the Clifford statistics. Using this statistics I proposed a simple model for the grand canonical ensemble of the carriers in the theory of fractional quantum Hall effect (FQHE). The model led to a temperature limit associated with the permutational degrees of freedom of such an ensemble.

As was pointed out before, building on the work on nonabelions of Read and Moore (1992) and Moore and Read (1990), Nayak and Wilczek (1996) and Wilczek (1998a,b) proposed a spinorial statistics for the FQHE carriers. The prototypical example was furnished by a so-called Pfaffian mode (occuring at filling fraction

$\nu = 1/2$), in which $2n$ quasiholes form a $2^{n-1}$-dimensional irreducible multiplet of the corresponding braid group. The new statistics was clearly nonabelian: it represented the permutation group $S_N$ on the $N$ individuals by a nonabelian group of operators in the $N$-body Hilbert space, a projective representation of $S_N$.

Since the subject is new, many unexpected effects in the systems of particles obeying Clifford statistics may arise in future experiments. One simple effect, which seems especially relevant to the FQHE, might be observed in a grand canonical ensemble of Clifford quasiparticles. Here I give its direct derivation first.

Following Read and Moore (1992) we postulate that only two quasiparticles at a time can be added to (or removed from) the FQHE ensemble. Thus, we start with an $N = 2n$-quasiparticle effective Hamiltonian whose only relevant (to our problem) energy level $E_{2n}$ is $2^{n-1}$-fold degenerate. The degeneracy of the ground mode with no quasiparticles present is taken to be $g(E_0) = 1$.

Assuming that adding a *pair* of quasiparticles to the composite increases the total energy by $\varepsilon$, and ignoring all the external degrees of freedom, we can tabulate the resulting many-body energy spectrum as follows:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Number of quasiparticles, $N = 2n$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 | $\cdots$ |
| Degeneracy, $g(E_{2n}) = 2^{n-1}$ | 0 | 1 | 2 | 4 | 8 | 16 | 32 | $\cdots$ |
| Composite energy, $E_{2n}$ | $0\varepsilon$ | $1\varepsilon$ | $2\varepsilon$ | $3\varepsilon$ | $4\varepsilon$ | $5\varepsilon$ | $6\varepsilon$ | $\cdots$ |

(G1)

Notice that the energy levels so defined furnish irreducible multiplets for projective representations of permutation groups in Schur's theory (Schur, 1911), as was first pointed out by Wilczek (1998a,b).

We now consider a grand canonical ensemble of Clifford quasiparticles.

The probability that the composite contains $n$ pairs of quasiparticles is

$$P(n, T) = \frac{g(E_{2n})\, e^{(n\mu - E_{2n})/k_B T}}{1 + \sum_{n=1}^{\infty} g(E_{2n})\, e^{(n\mu - E_{2n})/k_B T}}$$

$$\equiv \frac{2^{n-1}\, e^{-(n\mu - E_{2n})/k_B T}}{1 + \sum_{n=1}^{\infty} 2^{n-1}\, e^{-n(\varepsilon - \mu)/k_B T}}, \tag{G2}$$

where $\mu$ is the quasiparticle chemical potential. The denominator of this expression is the grand partition function of the composite

$$Z(T) = 1 + \sum_{n=1}^{\infty} g(E_{2n})\, e^{(n\mu - E_{2n})/k_B T}$$

$$\equiv 1 + \sum_{n=1}^{\infty} 2^{n-1}\, e^{-n(\varepsilon - \mu)/k_B T} \tag{G3}$$

at temperature $T$.

Now,

$$\sum_{n=1}^{\infty} 2^{n-1}\, e^{-nx} = e^{-x}[2^0\, e^{-0x} + 2^1\, e^{-1x} + 2^2\, e^{-2x} + \cdots] \\ = e^{-x} \sum_{n=0}^{\infty} e^{n(\ln 2 - x)}. \qquad \text{(G4)}$$

The partition function can therefore be written as

$$Z(T) = 1 + e^{-(\varepsilon-\mu)/k_B T} \sum_{n=0}^{\infty} e^{n(\ln 2 - (\varepsilon-\mu)/k_B T)}. \qquad \text{(G5)}$$

This leads to two interesting possibilities (assuming $\varepsilon > \mu$):

1) Regime $0 < T < T_c$, where

$$T_c = \frac{\varepsilon - \mu}{k_B\, \ln 2}. \qquad \text{(G6)}$$

Here the geometric series converges and

$$Z(T) = \frac{1 - e^{-(\varepsilon-\mu)/k_B T}}{1 - 2\, e^{-(\varepsilon-\mu)/k_B T}} = \frac{2\, e^{-(\varepsilon-\mu)/2k_B T}}{1 - 2\, e^{-(\varepsilon-\mu)/k_B T}} \sinh\left(\frac{\varepsilon-\mu}{2k_B T}\right). \qquad \text{(G7)}$$

The probability distribution is given by

$$P(n, T) = \frac{2^{n-1}\, e^{-n(\varepsilon-\mu)/k_B T}(1 - 2\, e^{-(\varepsilon-\mu)/k_B T})}{1 - e^{-(\varepsilon-\mu)/k_B T}}. \qquad \text{(G8)}$$

2) Regime $T \geq T_c$.

Under this condition the partition function diverges:

$$Z(T) = +\infty, \qquad \text{(G9)}$$

and the probability distribution vanishes:

$$P(n, T) = 0. \qquad \text{(G10)}$$

This result indicates that the temperature $T_c$ of (G6) is the upper bound of the intrinsic temperatures that the quasiparticle ensemble can have. Raising the temperature brings the system to higher energy levels which are more and more degenerate, resulting in a heat capacity that diverges at the temperature $T_c$.

To experimentally observe this effect, a FQHE system should be subjected to a condition where quasiparticles move freely between the specimen and a reservoir, without exciting other degrees of freedom of the system.

A similar limiting temperature phenomenon seems to occur in nature as the Hagedorn limit in particle physics (Hagedorn, 1968).

Knowing the partition function allows us to find various thermodynamic quantities of the quasiparticle system for subcritical temperatures $0 < T < T_c$. We are particularly interested in the average number of *pairs* in the grand ensemble:

$$\langle n \rangle_{\text{Cliff}} = \lambda \frac{\partial \ln Z}{\partial \lambda}, \tag{G11}$$

where $\lambda = e^{\mu/k_B T}$, or after some algebra,

$$\langle n(T) \rangle_{\text{Cliff}} = \frac{e^{-(\varepsilon-\mu)/k_B T}}{\left(1 - e^{-(\varepsilon-\mu)/k_B T}\right)\left(1 - 2\, e^{-(\varepsilon-\mu)/k_B T}\right)}. \tag{G12}$$

We can compare this with the familiar Bose–Einstein,

$$\langle n(T) \rangle_{\text{BE}} = \frac{1}{e^{(\varepsilon-\mu)/k_B T} - 1} \equiv \frac{e^{-(\varepsilon-\mu)/k_B T}}{1 - e^{-(\varepsilon-\mu)/k_B T}}, \tag{G13}$$

and Fermi–Dirac,

$$\langle n(T) \rangle_{\text{FD}} = \frac{1}{e^{(\varepsilon-\mu)/k_B T} + 1} \equiv \frac{e^{-(\varepsilon-\mu)/k_B T}}{1 + e^{-(\varepsilon-\mu)/k_B T}}, \tag{G14}$$

distributions. For the Clifford oscillator, $\langle n(T) \rangle_{\text{Cliff}} \to +\infty$ as $T \to T_c-$, as had to be expected.

To relate Schur's theory of projective representations of the permutation groups (descibed in Appendix F) to physics we may try to define a new, purely permutational variable of the Clifford composite, whose spectrum would reproduce the degeneracy of Read and Moore's theory.

A convenient way to define such a variable is by the process of quantification.

Let us thus assume that if there is just one quasiparticle in the system, then there is a limit on its localization, so that the quasiparticle can occupy only a finite number of sites in the medium, say $N = 2n$. We further assume that the Hilbert space of the quasiparticle is *real* and $N = 2n$-dimensional, and that a one-body variable (which upon quantification corresponds to the permutational variable of the ensemble) is an antisymmetric generator of an orthogonal transformation of the form

$$G := A \begin{bmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{bmatrix}, \tag{G15}$$

where $A$ is a constant coefficient. Note that in the complex case this operator would be proportional to the imaginary unit $i$, and the corresponding unitary transformation would be a simple multiplication by a phase factor with no observable effect. Since the quantified operator algebra for $N > 1$ quasiparticles will be complex, the effect of just one such "real" quasiparticle should be regarded as negligible in the grand canonical ensemble.

In the noninteracting case the process of quantification converts $G$ into a many-body operator $\hat{G}$ by the rule

$$\hat{G} := \sum_{l,j}^{N} \hat{e}_l G^l{}_j \hat{e}^j, \tag{G16}$$

where usually $\hat{e}_i$ and $\hat{e}^j$ are creators and annihilators, but in more general situations are the generators (that appear in the commutation or anticommutation relations) of the many-body operator algebra.

In Clifford statistics the generators of the algebra are Clifford units $\gamma_i = 2\hat{e}_i = -\gamma_i{}^\dagger$, and so quantification of $G$ proceeds as follows:

$$\begin{aligned} \hat{G} &= -A \sum_{k=1}^{n} \left(\hat{e}_{k+n}\hat{e}^k - \hat{e}_k\hat{e}^{k+n}\right) \\ &= +A \sum_{k=1}^{n} (\hat{e}_{k+n}\hat{e}_k - \hat{e}_k\hat{e}_{k+n}) \\ &= 2A \sum_{k=1}^{n} \hat{e}_{k+n}\hat{e}_k \\ &\equiv \frac{1}{2} A \sum_{k=1}^{n} \gamma_{k+n}\gamma_k. \end{aligned} \tag{G17}$$

Again, by Stone's theorem, the generator $\hat{G}$ acting on the spinor space of the complex Clifford composite of $N = 2n$ individuals can be factored into a Hermitian operator $\tilde{O}$ and an imaginary unit $i$ that commutes strongly with $\tilde{O}$:

$$\hat{G} = i\tilde{O}. \tag{G18}$$

We suppose that $\tilde{O}$ corresponds to the permutational many-body variable mentioned above, and seek its spectrum.

We note that $\hat{G}$ is a sum of $n$ commuting anti-Hermitian algebraically independent operators $\gamma_{k+n}\gamma_k$, $k = 1, 2, \ldots, n$, $(\gamma_{k+n}\gamma_k)^\dagger = -\gamma_{k+n}\gamma_k$, $(\gamma_{k+n}\gamma_k)^2 = -1$. If we now use $2^n \times 2^n$ complex matrix representation of Brauer and Weyl (F14) for the $\gamma$-matrices, we can simultaneously diagonalize the $2^n \times 2^n$ matrices representing the commuting operators $\gamma_{k+n}\gamma_k$, and use their eigenvalues, $\pm i$, to find the spectrum of $\hat{G}$, and consequently of $\tilde{O}$. The final result is obvious: there are $2^{2n}$ eigenkets of $\tilde{O}$, corresponding to the dimensionality of the spinor space of $\mathrm{Cliff}_{\mathbb{C}}(2n)$. In the irreducible representation of $S_N$ this number reduces to $2^{n-1}$, as required by Read and Moore's theory.

Note that in this approach the possible number of quasiholes in the ensemble is fixed by the number of the available sites, $N = 2n$. A change in that number must be accompanied by a change in the dimensionality of the one-quasiparticle

Hilbert space. It is natural to assume that variations in the physical volume of the entire system would provide such a mechanism.

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